

Math 285J, Assignment 5:

I. Consider the binary segmentation model based on the minimal variance in a level set formulation:

$$\inf_{c_1, c_2, \phi} F(c_1, c_2, \phi)$$

$$= \lambda_1 \int_{\Omega} (c_1 - f)^2 H(\phi) dx dy + \lambda_2 \int_{\Omega} (c_2 - f)^2 (1 - H(\phi)) dx dy + \mu \int_{\Omega} |\nabla H(\phi)| dx dy,$$

where $f : \Omega \rightarrow \mathbb{R}$ is the given image data to be segmented, $\lambda_1, \lambda_2, \mu > 0$ are tuning parameters (usually, we may take $\lambda_1 = \lambda_2 = 1$), and H is the 1D Heaviside function.

- For fixed level set function ϕ , find explicit expressions for c_1, c_2 minimizers of the functional F .

- For fixed c_1, c_2 , give the time-dependent Euler-Lagrange equation in ϕ , associated with the minimization. Thus $\phi = \phi(t, x, y)$ solves a PDE. Simplify this equation if possible by using cancellations (you may need to assume that the Heaviside function H has been substituted by a smooth approximation).

- Show that $F(c_1(t), c_2(t), \phi(t, \cdot, \cdot))$ is decreasing in time if c_1, c_2 and ϕ satisfy the obtained equations.

- Implement the time-dependent pde in ϕ together with the expressions for c_1 and c_2 and segment the image given on the class webpage. You need to start with an initial guess for the level set function (for example the signed distance function to a circle). Visualize the evolving curve (zero level line of ϕ) at intermediate steps and at steady state, and the binary segmented image given by $u = c_1 H(\phi) + c_2 (1 - H(\phi))$. Use a special approximation to the Heaviside function and to its derivative δ , given by

$$H_{\epsilon}(z) = \frac{1}{2} \left(1 + \frac{2}{\pi} \arctan \left(\frac{z}{\epsilon} \right) \right), \quad \delta_{\epsilon}(z) = H'_{\epsilon}(z).$$

Usually you can take $\epsilon = \Delta x = \Delta y = 1$. You can keep $\lambda_1 = \lambda_2 = 1$ and vary only μ to get the desired result. For more details, see the manuscript “Active contours without edges” from the class webpage.

II. Using the notations from the lecture, find a geometric time-dependent gradient descent evolution that minimizes the robust alignment term

$$E(C) := \int_0^1 \left| \langle V, \vec{N} \rangle \right| |C_q| dq,$$

where $V = (u, v)$ is a given vector field and $\vec{N} = \frac{(-\frac{\partial C_2}{\partial q}, \frac{\partial C_1}{\partial q})}{|C_q|}$ is the unit normal. Particularize this to the case $V = \nabla I$.