

Math 285J
Assignment 4:

[1] *Affine invariance:* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary matrix, such that $ad - bc > 0$, and let $X = (x, y)$. Check that, if u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| \text{curv}(u)^{1/3},$$

then $v(X) = u(AX)$ satisfies

$$\frac{\partial v}{\partial t} = c(A) |\nabla v| \text{curv}(v)^{1/3}.$$

What is $c(A)$?

[2] *Computational exercise:* Segmentation via the Ambrosio-Tortorelli elliptic approximations to the Mumford and Shah functional

Consider given image data $f \in L^\infty(\Omega) \subset L^2(\Omega)$. We wish to solve in practice the following minimization problem (where $\varepsilon \rightarrow 0^+$ is a small parameter)

$$\inf_{u,v \in H^1(\Omega)} G_\varepsilon^{AT}(u, v) = \int_\Omega \left[\varepsilon |\nabla v|^2 + \alpha(v^2 + o_\varepsilon) |\nabla u|^2 + \frac{(v-1)^2}{4\varepsilon} + \beta |u-f|^2 \right] dx dy, \quad (1)$$

where o_ε is any non negative infinitesimal faster than ε , and α, β are positive parameters. The unknown $u = u_\varepsilon$ will be an optimal piecewise-smooth approximation of the data f , while the unknown $v = v_\varepsilon$ will be an edge function: $0 \leq v \leq 1$, $v \approx 0$ near edges, and $v \approx 1$ inside homogenous regions.

(i) Give the Euler-Lagrange equations in u and v , associated with the minimization (using alternating minimization), together with the corresponding boundary conditions.

(ii) Discretize the obtained system and implement it for an image f . Visualize the energy decrease versus iterations, and the final u and v at steady state.