Assignment #3

[1] Given $f \in L^2(\Omega)$, consider the minimization problem

$$\inf_{u \in H^1(\Omega)} F(u) = \int_\Omega \lambda |f - Ku|^2 dxdy + \sqrt{\int_\Omega |\nabla u|^2 dxdy},$$

where $\lambda > 0$ and K is linear and continuous operator from $L^2(\Omega)$ to $L^2(\Omega)$, with adjoint K^*, such that $K1 = 1$.

Formulate and show a similar characterization of minimizers (as done in class for the BV ROF model). Define the dual star norm $\|\cdot\|_*$ necessary in this formulation and mention to what known norm this corresponds.

[2] Let $u : \mathbb{R}^2 \to \mathbb{R}$. The upper level set of an image function u at level $\lambda \in \mathbb{R}$ is the set $\chi_\lambda(u) = \{x \in \mathbb{R}^2 : \ u(x) \geq \lambda\}$. Show that u can be retrieved by the reconstruction formula

$$u(x) = sup\{\lambda : \ x \in \chi_\lambda(u)\}.$$

[3] Let $u, v : \mathbb{R}^2 \to \mathbb{R}$. Assume that two image functions u and v have the same level sets, that is for all $\lambda \in \mathbb{R}$, there is $\mu \in \mathbb{R}$ such that $\chi_\lambda(u) = \chi_\mu(v)$. Let us define g by $g(\lambda) = \sup\Big\{\mu : \ \chi_\lambda(u) = \chi_\mu(v)\Big\}$. Then g is nondecreasing and $v = g \circ u$. (show first that g is nondecreasing, then show that $v \geq g \circ u$ and that $g \circ u \geq v$).

[4] Implement the projection algorithm by A. Chambolle, introduced in the paper posted on the class web-page, for ROF total variation minimization (implement equation (9) from the paper, and obtain the denoised output image u using (7)). Compare with your implementation from the previous assignment.