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Assignment 2:

[**1**] Implement a numerical scheme for the Euler-Lagrange equation of the ROF model in the presence of known blur:

$$\inf_u \int_\Omega \sqrt{\epsilon^2 + |\nabla u|^2} dxdy + \frac{\lambda}{2} \int_\Omega |k * u - f|^2 dxdy,$$

where you can choose a Gaussian blur or a uniform blur for k (of mean 1), with small support (between 3×3 and 9×9). If you compute the convolution in the spatial domain, then you can use the commands "conv2" or "imfilter" in matlab. You can also evaluate the convolutions in the frequency domain. Define a blurry data f (with a small amount of noise). Output the root-mean-square-error between u and u_{orig}, and plot the numerical energy versus iterations. Choose a value λ that gives better results.

[**2**] Assume that $\phi(z)$ is even and differentiable, increasing on $[0, \infty)$ and positive. We know that the time-dependent Euler-Lagrange equation obtained by minimizing

$$\inf_{u \in W^{1,1}(\Omega)} F(u) = \int_\Omega \phi(|\nabla u|) dxdy + \frac{\lambda}{2} \int_\Omega |f - u|^2 dxdy$$

is formally given by

$$(1) \qquad \frac{\partial u}{\partial t} = -\lambda(u - f) + \mathrm{div}\Big(\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u\Big).$$

(i) Express the above differential operator in (1) as $()u_{\vec{N}\vec{N}} + ()u_{\vec{T}\vec{T}}$, where $\vec{N} = \frac{\nabla u}{|\nabla u|}$, and $\vec{T}$ is normalized and orthogonal to $\vec{N}$.

(ii) Assume $\lambda = 0$ in (1). Under what conditions is the obtained PDE (weakly) parabolic ? (in other words, under what conditions on ϕ, does the quasi-linear 2nd order operator $\mathrm{div}\Big(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\Big) = \sum_{i,j=1}^2 a_{ij} u_{x_i,x_j}$ satisfy the weakly elliptic property $\sum_{i,j=1}^2 a_{ij}\xi_i\xi_j \geq 0$ for all $\xi_1, \xi_2 \in R$?)

[**3**] Let u be sufficiently smooth and satisfy

$$\frac{\partial u}{\partial t} = |\nabla u| G(\mathrm{curv} u),$$

where $\mathrm{curv} u = \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)$ is the curvature operator, and G is a function such that $kG(k) \geq 0$. Show that this flow decreases the total variation of u in time.

[**4**] Let $g \in C^1(R)$ be a function, with $g' > 0$. Let $v = g(u)$. If u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| G(\mathrm{curv}(u)),$$

so does v (this is called contrast invariance or geometric invariance).