Nonlocal methods for image processing

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Outline

1. Local smoothing Filters
2. Nonlocal means filter
3. Nonlocal operators
4. Applications
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General Model

\[ v(x) = u(x) + n(x), \ x \in \Omega \]

- \( v(x) \) observed image
- \( u(x) \) true image
- \( n(x) \) i.i.d gaussian noise (white noise)

Gaussian kernel

\[ x \rightarrow G_h(x) = \frac{1}{4\pi h^2} e^{-\frac{|x|^2}{4h^2}} \]
Nonlocal methods for image processing
Local smoothing Filters

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Linear low-pass filter

Idea: average in a local spatial neighborhood

\[ GF_h(v)(x) = G_h * v(x) = \frac{1}{C(x)} \int_{y \in \Omega} v(y) \exp \left( \frac{||y-x||^2}{4h^2} \right) dy \]

where \( C(x) = 4\pi h^2 \)

Pro: work well for harmonic function (homogenous region)

Con: perform poorly on singular part, namely edge and texture
Anisotropic filter

Idea: average only in the direction orthogonal to $Dv(x)(\frac{\partial v(x)}{\partial x}, \frac{\partial v(y)}{\partial y})$.

$$AF_h(v)(x) = \frac{1}{C(x)} \int t v(x + f \frac{Dv(x) \perp}{|Dv(x)|}) \exp \frac{-t^2}{h^2} dt$$

where $C(x) = 4\pi h^2$.

Pro: Avoid blurring effect of Gaussian filter, maintaining edges.
Con: perform poorly on flat region, worse there than a Gaussian blur.
Neighboring filter

Spatial neighborhood

\[ B_\rho(x) = \{ y \in \Omega \| y - x \| \leq \rho \} \]

Gray-level neighborhood

\[ B(x, h) = \{ y \in \Omega \| v(y) - v(x) \| \leq \rho \} \]

for a given image \( v \). Yaroslavsky filter

\[ YNF_{h, \rho} = \frac{1}{C(x)} \int_{B_\rho(x)} u(y) e^{-\frac{|u(y) - u(x)|^2}{4h^2}} dy \]

Bilateral(SUSAN) filter

\[ SUSAN_{h, \rho} = \frac{1}{C(x)} \int u(y) e^{-\frac{|u(y) - u(x)|^2}{4h^2}} e^{-\frac{|y-x|^2}{4\rho^2}} dy \]

Behave like weighted heat equation, enhancing the edges
Fig. 3. Denoising experience on a natural image. From left to right and from top to bottom: noisy image (standard deviation 20), Gaussian convolution, anisotropic filter, total variation minimization, Tadmor-Nezzar-Vese iterated total variation, Osher et al. iterated total variation, and the Yaroslavsky neighborhood filter.
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Nonlocal mean filter

Idea: Take advantage of high degree of redundancy of natural images

Fig. 6. $q_1$ and $q_2$ have a large weight because their similarity windows are similar to that of $p$. On the other side the weight $w(p, q_3)$ is much smaller because the intensity grey values in the similarity windows are very different.
Denoising formula

\[ NLM(v)(x) := \frac{1}{C(x)} \int_{\Omega} w(x, y)v(y)dy, \]

where

\[ w(x, y) = \exp\left\{-\frac{G_{\alpha} \ast (\|v(x + \cdot) - v(y + \cdot)\|^2)(0)}{2h_0^2}\right\}, \]

\[ C(x) = \int_{\Omega} w_v(x, y)dy \]
Weight from clean image

(a) 

(b) 

(c) 

(d) 

(e) 

(f)
Weight from noisy image

(a) 
(b) 
(c) 
(d)
Example

Fig. 7. NL-means denoising experiment with a nearly periodic image. Left: Noisy image with standard deviation 30. Right: NL-means restored image.

Fig. 8. NL-means denoising experiment with a Brodatz texture image. Left: Noisy image with standard deviation 30. Right: NL-means restored image. The Fourier transform of the noisy and
Comparison with other methods

**Fig. 20.** Denoising experience on a natural image. From left to right and from top to bottom: noisy image (standard deviation 35), neighborhood filter, total variation, and the NL-means algorithm.
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Nonlocal operators\textsuperscript{2}/Graph based Regularization

Given a nonnegative and symmetric weight function \( w(x, y) \) for each pair of points \((x, y) \in \Omega \times \Omega\):

- Nonlocal gradient of an image \( u(x) \):
  \[
  \nabla_w u(x, y) = (u(y) - u(x)) \sqrt{w(x, y)} : \Omega \times \Omega \rightarrow \Omega
  \]

- Nonlocal divergence of a gradient field \( p(x, y) : \Omega \times \Omega \rightarrow \mathcal{R} \) is defined by
  \[
  \langle \nabla_w u, p \rangle = - \langle u, \text{div}_w p \rangle, \forall u(x), p(x, y)
  \]
  \[
  \Rightarrow \text{div}_w p(x) = \int_{\Omega} (p(x, y) - p(y, x)) \sqrt{w(x, y)} dy.
  \]

- Nonlocal functionals of \( u \):
  \[
  J_{NL/H^1}(f) = \frac{1}{4} \int_{\Omega} |\nabla_w u(x)|^2 : \frac{1}{4} \int_x \int_y |\nabla_w u(x, y)|^2
  \]
  \[
  J_{NL/TV}(f) = \int_{\Omega} |\nabla_w u(x)|_1 : \int_x \sqrt{\int_y |\nabla_w u(x, y)|^2}.
  \]
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Nonlocal $H^1$ regularization by non-local means

- Model: $\min J_{NL/H1}(u) + \frac{\mu}{2} \|u - f\|^2$
- Euler-Lagrange equation: $L_w(u)u + \mu(u - f) = 0$, where $L_w$ is unnormalized graph laplacian:

$$L_w(u) = \int_\Omega w(x,y)(u(x) - u(y)).$$

- We can replace $L_w(u)$ by normalized graph laplacian$^3$

$$L_0^w = \frac{1}{C(x)}L_w = Id - NLM_w(u).$$

- Semi-explicit iteration: for a time step

  $\tau > 0, s = 1 + \tau + \tau \mu, \alpha_1 = \frac{\tau}{s}, \alpha_2 = \frac{\tau \mu}{s}$:

$$u^{k+1} = (1 - \alpha_1)u^k + \alpha_1 NLM_w(u^k) + \alpha_2 f.$$

$^3$When $N \to \infty$ and $h_0 \to 0$, then $L_0^w$ converges to the continuous manifold Laplace - Beltrami operator.
Nonlocal TV regularization by Chambolle’s algorithm

- **Model:** \( \min_u J_{NL/TV,w}(u) + \frac{\mu}{2} ||u - f||^2 \)
- **Extension of Chambolle’s projection method for Nonlocal TV:**

\[
\inf_u \sup_{||p|| \leq 1} \int_{\Omega \times \Omega} < \nabla_w u, p > + \frac{\mu}{2} ||u - f||^2,
\]

where the solution can be solved by a projected solution \( u^* = f - \frac{1}{\mu} \div_w p^* \). and the dual variable \( p^* \) is obtained by

\[
\sup_{||p|| \leq 1} \int_{\Omega \times \Omega} < \nabla_w u, p > + \frac{1}{2\mu} ||\div_w p||^2.
\]

**Algorithm:**

\[
p^{n+1} = \frac{p^n + \tau \nabla_w (\div_w p^n - \mu f)}{1 + \tau |\nabla_w (\div_w p^n - \mu f)|}, \quad \tau > 0
\]
Deblurring by Nonlocal Means

Problem: \( f = Au + n \), \( A \) linear operator, \( n \) Gaussian noise. **Idea:** Use initial blurry and noisy image \( f \) to compute the weight.

\[
J_{\text{NLM},w(f)} := \min ||u - \text{NLM}_f u||^2 + \frac{\lambda}{2} ||Au - f||^2
\]  

which is equivalent to

\[
J_{\text{NLM},w(f)} := \min ||L^0_{w_f}(u)||^2 + \frac{\lambda}{2} ||Au - f||^2
\]

where \( L^0_{w_f} \) is the normalized graph laplacian with the weight computed from \( f \).

Gradient descents flow:

\[
((L^0_{w_f})^* L^0_{w_f})u + \lambda A^*(Au - f) = 0
\]

\(^4\)A. Buades, B. Coll, and J-M. Morel. 2006
**Image recovery via nonlocal operators**

**Idea:** Use a deblurred image to compute the weight.

1. **Preprocessing:**
   - Compute a deblurred image via a fast method:
     \[
     u_0 = \min \frac{1}{2} ||Au - f||^2 + \delta ||u||^2 \iff u_0 = (A^*A + \delta)^{-1} A^*f.
     \]
     where $\delta$ is chosen optimally by respecting the condition
     \[
     \sigma^2 = ||Au_0 - f||^2
     \]
     where $\sigma^2$ is the noise level in blurry image.
   - Compute the nonlocal weight $w_0$ by using $u_0$ as a reference image (set $h_0 = \sigma^2 ||(A^*A + \delta)^{-1} A^*||^2$.)

2. **Nonlocal regularization with the fixed weight $w_0$:**
   \[
   \min J_{w_0}(u) + \frac{\lambda}{2} ||Au - f||^2
   \]
   by gradient descent.
Nonlocal regularization for inverse problems

- **Idea:** nonlocal weight updating during nonlocal regularization by operator splitting.

- **Model:**
  \[
  \min_u J_w(u)(u) + \frac{\lambda}{2} ||Au - v||^2
  \]

**Approximated Algorithm:**

\[
\begin{align*}
  v^{k+1} &= u^k + \frac{1}{\mu} A^*(f - Au^k) \\
  w^{k+1} &= w(v^{k+1})(optional) \\
  u^{k+1} &= \text{arg min } J_{NL/TV,w^{k+1}} + \frac{\lambda\mu}{2}||u - v^{k+1}||^2
\end{align*}
\]

where \(u^{k+1}\) is solved by Chambolle's method for NLTV.
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Nonlocal regularization with Bregmanized methods

With/without weight updating:

Algorithm:

\[
\begin{align*}
    f^{k+1} &= f^k + f - Au^k \\
v^{k+1} &= u^k + \frac{1}{\mu} A^* (f^{k+1} - Au^k) \\
w^{k+1} &= w(v^{k+1}) \text{(optional)} \\
u^{k+1} &= \arg \min J_{NL/TV, w^{k+1}} + \frac{\lambda \mu}{2} ||u - v^{k+1}||^2
\end{align*}
\]
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Compressive sampling: \( Au = RFu \)

**Figure:** Data: 30% random Fourier measurements
Deconvolution: $Au = k \ast u$

True Image  
Blury and noisy Image

Fix weight  
Update weight

Figure: $9 \times 9$ box average blur kernel, $\sigma = 3$
Wavelet Inpainting: $Au = RWu$

**Figure:** Block loss (including low-low frequencies loss). For both TV and NLTV, the initial guess is the received image.
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