Math 285J, Assignment 5: Due on Friday, December 4, or during the week of finals (no later than Friday, December 11!).

I. Consider the binary segmentation model based on the minimal variance in a level set formulation:

\[ \inf_{c_1, c_2, \phi} F(c_1, c_2, \phi) = \lambda_1 \int_{\Omega} (c_1 - f)^2 H(\phi) dx dy + \lambda_2 \int_{\Omega} (c_2 - f)^2 (1 - H(\phi)) dx dy + \mu \int_{\Omega} |\nabla H(\phi)| dx dy, \]

where \( f : \Omega \to \mathbb{R} \) is the given image data to be segmented, \( \lambda_1, \lambda_2, \mu > 0 \) are tuning parameters (usually, we may take \( \lambda_1 = \lambda_2 = 1 \)), and \( H \) is the 1D Heaviside function.

• For fixed level set function \( \phi \), find explicit expressions for \( c_1, c_2 \) minimizers of the functional \( F \).
• For fixed \( c_1, c_2 \), give the time-dependent Euler-Lagrange equation in \( \phi \), associated with the minimization. Thus \( \phi = \phi(t, x, y) \) solves a PDE. Simplify this equation if possible by using cancellations (you may need to assume that the Heaviside function \( H \) has been substituted by a smooth approximation).
• Show that \( F(c_1(t), c_2(t), \phi(t, \cdot, \cdot)) \) is decreasing in time if \( c_1, c_2 \) and \( \phi \) satisfy the obtained equations.
• Implement the time-dependent pde in \( \phi \) together with the expressions for \( c_1 \) and \( c_2 \) and segment the image given on the class webpage. You need to start with an initial guess for the level set function (for example the signed distance function to a circle). Visualize the evolving curve (zero level line of \( \phi \)) at intermediate steps and at steady state, and the binary segmented image given by \( u = c_1(\phi) + c_2(1 - H(\phi)) \). Use a special approximation to the Heaviside function and to its derivative \( \delta \), given by

\[ H_\epsilon(z) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{z}{\epsilon} \right) \right), \quad \delta_\epsilon(z) = H'_\epsilon(z). \]

Usually you can take \( \epsilon = \triangle x = \triangle y = 1 \). You can keep \( \lambda_1 = \lambda_2 = 1 \) and vary only \( \mu \) to get the desired result. For more details, see the manuscript “Active contours without edges” from the class webpage.

II. Using the notations from the lecture, find a geometric time-dependent gradient descent evolution that minimizes the robust alignment term

\[ E(C) := \int_0^1 \left| \langle V, \vec{N} \rangle \right| |C_q| dq, \]

where \( V = (u, v) \) is a given vector field and \( \vec{N} = \left( \frac{\partial c_2}{\partial q}, \frac{\partial c_1}{\partial q} \right) / |C_q| \) is the unit normal. Particularize this to the case \( V = \nabla I \).