**Math 285J, Assignment 5:** Due on Friday, December 4, or during the week of finals (no later than Friday, December 11 !).

**I.** Consider the binary segmentation model based on the minimal variance in a level set formulation:

$$\inf_{c_1,c_2,\phi} F(c_1,c_2,\phi)$$

$$= \lambda_1 \int_\Omega (c_1-f)^2 H(\phi)dxdy + \lambda_2 \int_\Omega (c_2-f)^2(1-H(\phi))dxdy + \mu \int_\Omega |\nabla H(\phi)|dxdy,$$

where $f:\Omega \to I\!\!R$ is the given image data to be segmented, $\lambda_1, \lambda_2, \mu > 0$ are tuning parameters (usually, we may take $\lambda_1 = \lambda_2 = 1$), and $H$ is the 1D Heaviside function.

• For fixed level set function $\phi$, find explicit expressions for $c_1$, $c_2$ minimizers of the functional $F$.

• For fixed $c_1$, $c_2$, give the time-dependent Euler-Lagrange equation in $\phi$, associated with the minimization. Thus $\phi = \phi(t,x,y)$ solves a PDE. Simplify this equation if possible by using cancellations (you may need to assume that the Heaviside function $H$ has been substituted by a smooth approximation).

• Show that $F(c_1(t), c_2(t), \phi(t,\cdot,\cdot))$ is decreasing in time if $c_1$, $c_2$ and $\phi$ satisfy the obtained equations.

• Implement the time-dependent pde in $\phi$ together with the expressions for $c_1$ and $c_2$ and segment the image given on the class webpage. You need to start with an initial guess for the level set function (for example the signed distance function to a circle). Visualize the evolving curve (zero level line of $\phi$) at intermediate steps and at steady state, and the binary segmented image given by $u = c_1 H(\phi) + c_2(1-H(\phi))$. Use a special approximation to the Heaviside function and to its derivative $\delta$, given by

$$H_\epsilon(z) = \frac{1}{2}\Big(1 + \frac{2}{\pi}\arctan\Big(\frac{z}{\epsilon}\Big)\Big), \quad \delta_\epsilon(z) = H'_\epsilon(z).$$

Usually you can take $\epsilon = \triangle x = \triangle y = 1$. You can keep $\lambda_1 = \lambda_2 = 1$ and vary only $\mu$ to get the desired result. For more details, see the manuscript "Active contours without edges" from the class webpage.

**II.** Using the notations from the lecture, find a geometric time-dependent gradient descent evolution that minimizes the robust alignment term

$$E(C) := \int_0^1 \Big|\Big\langle V, \vec{N}\Big\rangle\Big| |C_q| dq,$$

where $V = (u,v)$ is a given vector field and $\vec{N} = \frac{(-\frac{\partial C_2}{\partial q}, \frac{\partial C_1}{\partial q})}{|C_q|}$ is the unit normal. Particularize this to the case $V = \nabla I$.