Math 285J
Assignment 4: Due on Wednesday, November 25

[**1**] *Affine invariance:* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary matrix, such that $ad - bc > 0$, and let $X = (x, y)$. Check that, if u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| \mathrm{curv}(u)^{1/3},$$

then $v(X) = u(AX)$ satisfies

$$\frac{\partial v}{\partial t} = c(A) |\nabla v| \mathrm{curv}(v)^{1/3}.$$

What is $c(A)$?

[**2**] *Computational exercise:* Segmentation via the Ambrosio-Tortorelli elliptic approximations to the Mumford and Shah functional

Consider given image data $f \in L^\infty(\Omega) \subset L^2(\Omega)$. We wish to solve in practice the following minimization problem (where $\varepsilon \to 0^+$ is a small parameter)

$$\inf_{u,v \in H^1(\Omega)} G_\varepsilon^{AT}(u, v) = \int_\Omega \Big[\varepsilon |\nabla v|^2 + \alpha (v^2 + o_\varepsilon) |\nabla u|^2 + \frac{(v-1)^2}{4\varepsilon} + \beta |u - f|^2 \Big] dxdy, \tag{1}$$

where o_ε is any non negative infinitesimal faster than ε, and α, β are positive parameters. The unknown $u = u_\varepsilon$ will be an optimal piecewise-smooth approximation of the data f, while the unknown $v = v_\varepsilon$ will be an edge function: $0 \leq v \leq 1$, $v \approx 0$ near edges, and $v \approx 1$ inside homogenous regions.

(i) Give the Euler-Lagrange equations in u and v, associated with the minimization (using alternating minimization), together with the corresponding boundary conditions.

(ii) Discretize the obtained system and implement it for an image f. Visualize the energy decrease versus iterations, and the final u and v at steady state.