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**Assignment 3:** Due on Monday, November 2

[1] Implement a numerical scheme for the Euler-Lagrange equation of the ROF model in the presence of known blur:

$$\inf_u \int_{\Omega} \sqrt{\epsilon^2 + |\nabla u|^2} dx dy + \frac{\lambda}{2} \int_{\Omega} |k * u - f|^2 dx dy,$$

where you can choose a Gaussian blur or a uniform blur for  $k$  (of mean 1), with small support (between  $3 \times 3$  and  $9 \times 9$ ). If you compute the convolution in the spatial domain, then you can use the commands “conv2” or “imfilter” in matlab. You can also evaluate the convolutions in the frequency domain. Define a blurry data  $f$  (with a small amount of noise). Output the root-mean-square-error between  $u$  and  $u_{orig}$ , and plot the numerical energy versus iterations. Choose a value  $\lambda$  that gives better results.

[2] Assume that  $\phi(z)$  is even and differentiable, increasing on  $[0, \infty)$  and positive. We know that the time-dependent Euler-Lagrange equation obtained by minimizing

$$\inf_{u \in W^{1,1}(\Omega)} F(u) = \int_{\Omega} \phi(|\nabla u|) dx dy + \frac{\lambda}{2} \int_{\Omega} |f - u|^2 dx dy$$

is formally given by

$$(1) \quad \frac{\partial u}{\partial t} = -\lambda(u - f) + \operatorname{div} \left( \frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right).$$

(i) Express the above differential operator in (1) as  $(\cdot)_{\vec{N}\vec{N}} + (\cdot)_{\vec{T}\vec{T}}$ , where  $\vec{N} = \frac{\nabla u}{|\nabla u|}$ , and  $\vec{T}$  is normalized and orthogonal to  $\vec{N}$ .

(ii) Assume  $\lambda = 0$  in (1). Under what conditions is the obtained PDE (weakly) parabolic? (in other words, under what conditions on  $\phi$ , does the quasi-linear 2nd order operator  $\operatorname{div} \left( \frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right) = \sum_{i,j=1}^2 a_{ij} u_{x_i, x_j}$  satisfy the weakly elliptic property  $\sum_{i,j=1}^2 a_{ij} \xi_i \xi_j \geq 0$  for all  $\xi_1, \xi_2 \in \mathbb{R}$ ?)

[3] Let  $u$  be sufficiently smooth and satisfy

$$\frac{\partial u}{\partial t} = |\nabla u| G(\operatorname{curv} u),$$

where  $\operatorname{curv} u = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right)$  is the curvature operator, and  $G$  is a function such that  $kG(k) \geq 0$ . Show that this flow decreases the total variation of  $u$  in time.

[4] Let  $g \in C^1(\mathbb{R})$  be a function, with  $g' > 0$ . Let  $v = g(u)$ . If  $u$  satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| G(\operatorname{curv}(u)),$$

so does  $v$  (this is called contrast invariance or geometric invariance).