Math 285J, L. Vese. **Assignment 1:** Due on Monday, October 19
(late homework accepted)

[1] Consider in two dimensions the functional minimization problem

\[
\inf_u F(u) = F_1(u) + \lambda F_2(u_0 - Ku),
\]

where \(u_0 : \Omega \to \mathbb{R}\) is a given degraded version of a true (unknown) image \(u : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^2\), and \(K\) is a linear and continuous operator on \(L^2(\Omega)\). Here, \(F_1\) represents the regularization term while \(F_2\) represents the data fidelity term. Recall that \(|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}\).

Assume \(u_0, u \in L^2(\Omega)\), \(F_1(u) = \int_\Omega \phi_1(|\nabla u|)dxdy\), \(F_2(u_0 - Ku) = \int_\Omega \phi_2(u_0 - Ku)dxdy\), where \(\nabla u = (u_x, u_y)\) is the spatial gradient operator, \(\phi_i : \mathbb{R} \to \mathbb{R}\) are functions of class \(C^1\) \((i = 1, 2)\), and that \(\phi_2(u_0 - Ku) \in L^2(\Omega)\), as long as \(u_0 - Ku \in L^2(\Omega)\).

(i) For \(u \in W^{1,1}(\Omega)\), obtain the Euler-Lagrange equation associated with the minimization problem in \(u\), in the stationary and time-dependent cases, together with the appropriate boundary conditions in \(u\) on \(\partial\Omega\). For the time-dependent case, show that the energy \(E(t) = F(u(x,y,t))\) is decreasing in time. \(^1\)

(ii) Show that, if \(\phi_i, i = 1, 2\) are both convex, and \(\phi_1\) is in addition non-decreasing from \([0, \infty)\) to \([0, \infty)\), then the functional \(F(u)\) is convex.

[2] Consider in two dimensions \(f \in L^2(\Omega)\), and \(u(\cdot, \lambda)\) the minimizer of

\[
F(u) = \lambda \int_\Omega |\nabla u|dxdy + \frac{1}{2} \int_\Omega (u - f)^2dxdy,
\]

with \(\lambda > 0\). Recall that \(|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}\) can be made differentiable substituting it by \(\sqrt{\epsilon^2 + (u_x)^2 + (u_y)^2}\).

(i) Using the result from the previous problem, give the associated Euler-Lagrange equation with the corresponding boundary conditions for a minimizer \(u = u(\cdot, \lambda) \in W^{1,1}(\Omega)\).

(ii) Show that the \(L^2\)-norm of \(u(\cdot, \lambda)\), given by \(\sqrt{\int_\Omega (u(x,y,\lambda))^2dxdy}\) is bounded by a constant independent of \(\lambda\).

(iii) Show (e.g. using the obtained stationary E.-L. equation and associated boundary condition), that

\[
\int_\Omega u(x,y,\lambda)dxdy = \int_\Omega f(x,y)dxdy.
\]

(iv) Show that \(u(\cdot, \lambda)\) converges in the \(L^1(\Omega) - strong\) topology to the average of the initial data. In other words, show that

\[
\lim_{\lambda \to \infty} \int_\Omega \left| u(x,y,\lambda) - \frac{\int_\Omega f(x,y)dxdy}{|\Omega|} \right|dxdy = 0.
\]

\(^1\)We may need to formally assume, in addition, that \((Ku)_t = K(u_t)\); this is natural for a linear and continuous operator \(K\) that does not depend on \(t\), for instance if \(Ku = ku\) and \(k = k(x,y)\) does not depend on \(t\).
Discretize and implement the stationary or the non-stationary E.-L. equation from [2] by the method of your choice using finite differences, for the denoising case. More details will be discussed in class. Choose an original true image \( \hat{u} \), and define a noisy version \( f = \hat{u} + \text{noise} \) (see matlab sample codes on the class web-page, or in matlab you can add noise of zero mean to an image using “imnoise”). Give the optimal \( \lambda \) (may be different in each case) and the RMSE between the original clean image \( \hat{u} \) and the reconstructed image \( u \):

\[
RMSE = \sqrt{\frac{\sum_{i=1}^{M} \sum_{j=1}^{N} (\hat{u}(i,j) - u(i,j))^2}{MN}}.
\]

Plot the energy versus iterations.

Optional: You can make additional tests by substituting the data fidelity term \( \|f - u\|^2_{L^2(\Omega)} \) above by \( \|f - u\|_{L^1(\Omega)} \) or by \( \|f - u\|_{L^1(\Omega)} \), and compare the results. Each method may require different \( \lambda \) for the same image. \( \lambda \) can also be automatically selected if we know the noise variance in the form \( \|f - u\|^2 = \sigma^2 \). Using a norm, instead of the norm square for the data fidelity term avoids the intensity loss drawback of the ROF model.