Numerical discretization of the Rudin-Osher-Fatemi model

We discretize (using a fixed point finite differences scheme) the Euler-Lagrange equation associated with the minimization of the total variation model of Rudin-Osher-Fatemi.

We would like to find the (unique) minimizer, u, of

$$\inf_u F(u) = \lambda \int_\Omega |f-u|^2 dxdy + \int_\Omega |\nabla u| dxdy,$$

where f is the noisy data and $\lambda > 0$ is a scaling parameter. The associated Euler-Lagrange equation of the Rudin-Osher-Fatemi model formally is

$$\begin{cases} u = f + \frac{1}{2\lambda} \text{ div } \left(\frac{\nabla u}{|\nabla u|}\right) \text{ in } \Omega \\ \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega. \end{cases}$$

First, we remove the singularity when $|\nabla u| = 0$, by approximating $F(u)$ by $F_\epsilon(u)$, where

$$F_\epsilon(u) = \lambda \int_\Omega |f-u|^2 dxdy + \int_\Omega \sqrt{\epsilon^2 + |\nabla u|^2} dxdy,$$

with $\epsilon > 0$ a small parameter. Then, the Euler-Lagrange equation minimizing $F_\epsilon(u)$ formally is:

$$u = f + \frac{1}{2\lambda}\text{div}\Big(\frac{\nabla u}{\sqrt{\epsilon^2 + |\nabla u|^2}}\Big) \text{ in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega. \tag{2}$$

Assume for simplicity $\Omega = (0,1)^2$, $h > 0$ and let $x_i = ih$, $y_j = jh$, $h = 1/M$, for $0 \leq i, j \leq M$, be the discrete points (in our numerical calculations, we have $h = 1$). We recall the following notations:

$$\begin{aligned} u_{i,j} &\approx u(x_i, y_j), \\ f_{i,j} &\approx f(x_i, y_j), \\ \triangle_\pm^x u_{i,j} &= \pm(u_{i\pm1,j} - u_{i,j}), \\ \triangle_\pm^y u_{i,j} &= \pm(u_{i,j\pm1} - u_{i,j}), \\ \triangle_0^x u_{i,j} &= (u_{i+1,j} - u_{i-1,j})/2, \text{ and} \\ \triangle_0^y u_{i,j} &= (u_{i,j+1} - u_{i,j-1})/2. \end{aligned}$$

A discrete form of the Euler-Lagrange equation is:

$$\begin{aligned}
u_{i,j} &= f_{i,j} + \frac{1}{2\lambda h}\triangle_-^x\left[\frac{1}{\sqrt{\epsilon^2 + (\frac{\triangle_+^x u_{i,j}}{h})^2 + (\frac{\triangle_0^y u_{i,j}}{h})^2}}\frac{\triangle_+^x u_{i,j}}{h}\right] \\
&\quad + \frac{1}{2\lambda h}\triangle_-^y\left[\frac{1}{\sqrt{\epsilon^2 + (\frac{\triangle_0^x u_{i,j}}{h})^2 + (\frac{\triangle_+^y u_{i,j}}{h})^2}}\frac{\triangle_+^y u_{i,j}}{h}\right] \\
&= f_{i,j} + \frac{1}{2\lambda h^2}\frac{u_{i+1,j} - u_{i,j}}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j}-u_{i,j}}{h})^2 + (\frac{u_{i,j+1}-u_{i,j-1}}{2h})^2}} \\
&\quad - \frac{1}{2\lambda h^2}\frac{u_{i,j} - u_{i-1,j}}{\sqrt{\epsilon^2 + (\frac{u_{i,j}-u_{i-1,j}}{h})^2 + (\frac{u_{i-1,j+1}-u_{i-1,j-1}}{2h})^2}} \\
&\quad + \frac{1}{2\lambda h^2}\frac{u_{i,j+1} - u_{i,j}}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j}-u_{i-1,j}}{2h})^2 + (\frac{u_{i,j+1}-u_{i,j}}{h})^2}} \\
&\quad - \frac{1}{2\lambda h^2}\frac{u_{i,j} - u_{i,j-1}}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j-1}-u_{i-1,j-1}}{2h})^2 + (\frac{u_{i,j}-u_{i,j-1}}{h})^2}}.
\end{aligned}$$

We use a fixed point Gauss-Seidel iteration method for the above equation and so we now introduce the following linearized equation:

$$\begin{aligned}
u_{i,j}^{n+1} &= f_{i,j} + \frac{1}{2\lambda h^2}\frac{u_{i+1,j}^n - u_{i,j}^{n+1}}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j}^n-u_{i,j}^n}{h})^2 + (\frac{u_{i,j+1}^n-u_{i,j-1}^n}{2h})^2}} \\
&\quad - \frac{1}{2\lambda h^2}\frac{u_{i,j}^{n+1} - u_{i-1,j}^n}{\sqrt{\epsilon^2 + (\frac{u_{i,j}^n-u_{i-1,j}^n}{h})^2 + (\frac{u_{i-1,j+1}^n-u_{i-1,j-1}^n}{2h})^2}} \\
&\quad + \frac{1}{2\lambda h^2}\frac{u_{i,j+1}^n - u_{i,j}^{n+1}}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j}^n-u_{i-1,j}^n}{2h})^2 + (\frac{u_{i,j+1}^n-u_{i,j}^n}{h})^2}} \\
&\quad - \frac{1}{2\lambda h^2}\frac{u_{i,j}^{n+1} - u_{i,j-1}^n}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j-1}^n-u_{i-1,j-1}^n}{2h})^2 + (\frac{u_{i,j}^n-u_{i,j-1}^n}{h})^2}}.
\end{aligned}$$

Introducing the notations:

$$\begin{aligned}
c_1 &= \frac{1}{\sqrt{\epsilon^2 + (\frac{u_{i+1,j}^n-u_{i,j}^n}{h})^2 + (\frac{u_{i,j+1}^n-u_{i,j-1}^n}{2h})^2}}, \\
c_2 &= \frac{1}{\sqrt{\epsilon^2 + (\frac{u_{i,j}^n-u_{i-1,j}^n}{h})^2 + (\frac{u_{i-1,j+1}^n-u_{i-1,j-1}^n}{2h})^2}},
\end{aligned}$$

$$
\begin{aligned}
c_3 &= \frac{1}{\sqrt{\epsilon^2 + (\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h})^2 + (\frac{u^n_{i,j+1}-u^n_{i,j}}{h})^2}}, \\
c_4 &= \frac{1}{\sqrt{\epsilon^2 + (\frac{u^n_{i+1,j-1}-u^n_{i-1,j-1}}{2h})^2 + (\frac{u^n_{i,j}-u^n_{i,j-1}}{h})^2}},
\end{aligned}
$$

and solving for $u^{n+1}_{i,j}$, we obtain:

$$
\begin{aligned}
u^{n+1}_{i,j} &= \Big(\frac{1}{1+\frac{1}{2\lambda h^2}(c_1+c_2+c_3+c_4)}\Big) \\
&\cdot \Big[f_{i,j} + \frac{1}{2\lambda h^2}(c_1 u^n_{i+1,j} + c_2 u^n_{i-1,j} + c_3 u^n_{i,j+1} + c_4 u^n_{i,j-1})\Big].
\end{aligned}
$$

We let $u^0_{i,j} = f_{i,j}$. Then, we note that if $m_1 \leq f_{i,j} \leq m_2$, for any $0 \leq i,j \leq M$, we have $m_1 \leq u^n_{i,j} \leq m_2$, for any $n \geq 0$. We use the above equation for $u^{n+1}_{i,j}$ for all interior points (x_i, y_j) such that $1 \leq i,j \leq M-1$.

The boundary condition can be implemented in the following way: if $u^n_{i,j}$ has been computed using the above numerical scheme for $1 \leq i,j \leq M-1$, then we let $u^n_{0,j} = u^n_{1,j}$, $u^n_{M,j} = u^n_{M-1,j}$, $u^n_{i,0} = u^n_{i,1}$, $u^n_{i,M} = u^n_{i,M-1}$, and $u^n_{0,0} = u^n_{1,1}$, $u^n_{0,M} = u^n_{1,M-1}$, $u^n_{M,0} = u^n_{M-1,1}$, $u^n_{M,M} = u^n_{M-1,M-1}$.

• The coefficient λ has to be optimized for each image. Too small λ will introduce too much smoothing in the recovered image u. However, too large λ will keep noise in the solution u.

• Note that this scheme may introduce some asymmetry, but not visible in general. Other schemes can be proposed, for instance alternating at each iteration the discretization of the div operator, with all four (schematic) choices

$\triangle^x_-(\triangle^x_+)$, $\triangle^y_-(\triangle^y_+)$ (the above scheme)
$\triangle^x_+(\triangle^x_-)$, $\triangle^y_+(\triangle^y_-)$
$\triangle^x_+(\triangle^x_-)$, $\triangle^y_-(\triangle^y_+)$
$\triangle^x_-(\triangle^x_+)$, $\triangle^y_+(\triangle^y_-)$