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Assignment 2: Due on Friday, May 13 (late homework accepted)

Consider in two dimensions $f \in L^2(\Omega)$, and $u(\cdot, \lambda)$ the unique minimizer of

$$F(u) = \lambda \int_\Omega |Du| dx + \frac{1}{2} \int_\Omega |u - f|^2 dx,$$

with $\lambda \geq 0$.

We know that $u(\cdot, \lambda)$ necessarily satisfies the Euler-Lagrange equation (formally)

$$\text{(E-L)} \qquad \begin{cases} u(x, \lambda) - f(x) = \lambda \text{div}\left(\frac{Du(x,\lambda)}{|Du(x,\lambda)|}\right) \text{ in } \Omega, \\ \frac{Du(x,\lambda)}{|Du(x,\lambda)|} \cdot \vec{n}(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\vec{n}(x)$ denotes the exterior unit normal to $\partial\Omega$ at the point $x \in \partial\Omega$.

In the above relations, assume for simplicity that all expressions are well defined, that $u(\cdot, \lambda)$ is a Sobolev function (for instance in $W^{1,1}(\Omega) \cap L^2(\Omega)$), and that $Du(x, \lambda) = \nabla u(x, \lambda)$ denotes the usual spatial gradient (in the distributional sense, and not as a measure).

Problems:

(1) Show that the L^2-norm of $u(\cdot, \lambda)$, given by $\sqrt{\int_\Omega (u(x, \lambda))^2 dx}$ is bounded by a constant independent of λ.

(2) Show, using the above E-L equation and above B.C., that

$$\int_\Omega u(x, \lambda) dx = \int_\Omega f(x) dx.$$

(3) Show that $u(\cdot, \lambda)$ converges in the $L^1(\Omega) - \textit{strong}$ topology to the average of the initial data. In other words, show that

$$\lim_{\lambda \to \infty} \int_\Omega \left| u(x, \lambda) - \frac{\int_\Omega f(x) dx}{|\Omega|} \right| dx = 0.$$