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Assignment 1: Due on Monday, April 18 (late homework accepted)

Consider in two space dimensions the functional minimization problem

$$\inf_u F(u) = F_1(u) + \lambda F_2(u_0 - Ku),$$

where $u_0 : \Omega \to R$ is a degraded version of a true (unknown) image $u : \Omega \to R$, $\Omega \subset R^2$, and K is a linear and continuous operator on $L^2(\Omega)$. Here, F_1 represents the regularization term while F_2 represents the fidelity term. Recall that $|\nabla u| = \sqrt{(u_x)^2 + (u_y)^2}$.

(a) Assume $u_0, u \in L^2(\Omega)$, $F_1(u) = \int_\Omega \phi_1(|\nabla u|)dxdy$, $F_2(u_0 - Ku) = \int_\Omega \phi_2(u_0 - Ku)dxdy$, where $\nabla u = (u_x, u_y)$ is the spatial gradient operator, $\phi_i : R \to R$ are functions of class C^1 $(i = 1, 2)$, and that $\phi_2'(u_0 - Ku) \in L^2(\Omega)$, as long as $u_0 - Ku \in L^2(\Omega)$.

(i) Obtain the Euler-Lagrange equation associated with the minimization problem in u, in the stationary and time-dependent cases, together with the appropriate boundary conditions in u on $\partial\Omega$. For the time-dependent case, show that the energy $E(t) = F(u(x, y, t))$ is decreasing in time. [1]

(ii) Show that, if ϕ_i, $i = 1, 2$ are both convex, and ϕ_1 is in addition non-decreasing from $[0, \infty)$ to $[0, \infty)$, then the functional $F(u)$ is convex (use the definition of a convex function).

(b) Computational exercises:

With the above notations, implement the stationary or the non-stationary E-L equation by the method of your choice (in any language) in the denoising cases:

(i) $K =$ identity operator, $F_1(u) = \int_\Omega |\nabla u| dxdy$, $F_2(u) = \int_\Omega (u_0 - u)^2 dxdy$.

(ii) $K =$ identity operator, $F_1(u) = \int_\Omega |\nabla u| dxdy$, $F_2(u) = \sqrt{\int_\Omega (u_0 - u)^2 dxdy}$.

Use the attached noisy image, and plot the energy versus iterations in each case.

Give the optimal λ (may be different in each case) and the RMSE between the original clean image $\hat{u}$ and the reconstructed image u:

$$RMSE = \sqrt{\frac{\sum_{i=1,j=1}^{i=M,j=N} (\hat{u}(i,j) - u(i,j))^2}{MN}}.$$

Give the E-L equations in the two cases (note that the second case is not exactly of the general form given above). You may need to regularize the functions that appear at the origin, where these may be non-differentiable.

[1] We may need to formally assume, in addition, that $(Ku)_t = K(u_t)$; this is natural for a linear and continuous operator K that does not depend on t, for instance if $Ku = k * u$ and $k = k(x, y)$ does not depend on t.