October 5, 2001 Problems set I (for enrolled students)
Due: in two weeks.

Let $(x, y) \mapsto u(x, y)$ be an image function (assumed of sufficient regularity).
• Verify that

$$\nabla\Big(\frac{\nabla u}{|\nabla u|}\Big) = \frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}.$$

This is the curvature operator, denoted by $\mathrm{curv}(u)$.
• Let $g \in C^1(R)$ be a function, with $g' > 0$. Let $v = g(u)$. If u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| G(\mathrm{curv}(u)),$$

so does v (contrast invariance).

• Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix, such that $ad - bc > 0$, and let $X = (x, y)$. Check that, if u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| \mathrm{curv}(u)^{1/3},$$

then $v(X) = u(AX)$ satisfies

$$\frac{\partial v}{\partial t} = c(A)|\nabla v| \mathrm{curv}(v)^{1/3}.$$

What is $c(A)$? (affine invariance).

• Recall the definition of a convex function. Show that, if $t \mapsto f(t)$ is convex, positive and nondecreasing in $[0, \infty)$, then the functional $u \mapsto F(u)$ defined by

$$F(u) = \int_\Omega (u_0 - Ku)^2 dxdy + \alpha \int_\Omega f(|\nabla u|) dxdy$$

is also convex, where $u_0(x, y)$ is a given function and $K : L^2(\Omega) \to L^2(\Omega)$ is a linear continuous operator.

• Assume that f from the previous exercise is of class C^2. Find (formally) the Euler-Lagrange equation associated with the minimization

$$\inf_u F(u),$$

by computing

$$\lim_{\epsilon \to 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon},$$

where v is a test function (use integration by parts, and the definition of the adjoint operator K^* of K). Give the obtained formula in divergence form and in two different expanded forms (one using u_{xx}, u_{xy}, u_{yy}, and the other one using u_{NN}, u_{TT}).