Math 273b: Calculus of Variations
Homework #3, due on Friday, June 3rd

[**1**] Consider the 1D length functional minimization problem

$$\min_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \min_u \int_0^1 \sqrt{1 + (u'(x))^2}dx,$$

for twice differentiable functions $u : [0, 1] \to I\!R$ with boundary conditions $u(0) = 0$, $u(1) = 1$.

(a) Show that the functional $u \mapsto F(u)$ is convex.

(b) Formally compute the Gateaux-differential and then obtain the Euler-Lagrange equation associated with the minimization.

(c) Find the exact solution of the problem.

[**2**] Let $A : I\!R^n \to I\!R^n$ be a linear self-adjoint operator, $b \in I\!R^n$, and consider the quadratic function for $x \in I\!R^n$

$$x \mapsto q(x) := \langle Ax, x\rangle - 2\langle b, x\rangle.$$

Show that the three statements

(i) $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$

(ii) $A \geq O$ and $b \in \mathrm{Im}A$.

(iii) the problem $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$ has a solution

are equivalent. When they hold, characterize the set of minimum points of q, in terms of the pseudo-inverse of A.

[**3**] Computation of the Euler-Lagrange equation.

(a) Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$ given constants, and L a sufficiently smooth function. Obtain formally the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal u.

Hint: Consider smooth test functions v, such that $v(x_0) = v(x_1) = 0$. Since u is a minimizer, we must have $F(u) \leq F(u+\varepsilon v)$ for all such sufficiently smooth functions v and every real ϵ. Apply integration by parts to obtain the desired result. You should obtain a second-order differential equation.

(b) Let now $u(x,t)$ be a smooth solution of the time-dependent partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u'}(x,u,u') - L_u(x,u,u'),$$

with $u(x,0) = u_0(x)$ on (x_0, x_1) and $u(x_0,t) = U_0$, $u(x_1,t) = U_1$ for $t \geq 0$. Show that the function $E(t) = F(u(\cdot,t))$ is decreasing in time, where $F(u) = \int_{x_0}^{x_1} L(x,u,u')dx$.

Optional problems

[**1**] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$, $u'(x_0) = U_0$, $u'(x_1) = U_1$ given, and L a sufficiently smooth function. As in the previous problem, derive the equation satisfied by a smooth optimal u. Choose test functions v in $C^\infty[x_0, x_1]$ that satisfy $v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0$. (you should obtain a fourth-order differential equation).

[**2**] Consider the minimization problem in two dimensions (x, y),

$$\inf_u E(u) = \int_\Omega L(x,y,u,u_x,u_y)dxdy, \quad u = g \text{ on } \partial\Omega,$$

where g is a given function on the boundary $\partial\Omega$, with Ω a bounded and open region in the plane. Assume that the integrand L is differentiable.

(i) Show that a sufficiently smooth minimizer u formally satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P) = 0$$

on Ω, where $P = (x, y, u(x,y), u_x(x,y), u_y(x,y))$.

(ii) Apply the above result to the case when $L(x,y,u_x,u_y) = u_x^2 + u_y^2 - 2fu$.

Hint for (i): consider another test function v, such that $v = 0$ on $\partial\Omega$. Since u is a minimizer, we must have $E(u) \leq E(u + \varepsilon v)$ for all such sufficiently smooth functions v and all real ϵ. Apply integration by parts to obtain the desired result. Here, $(u_x, u_y) = \nabla u$.

Notes:

• **Pseudo-Inverse.** If A is a symmetric (or self-adjoint) linear operator on X, then $\mathrm{Im}A^{\perp} = \mathrm{Ker}A$. Let $p_{\mathrm{Im}A}$ be the operator of orthogonal projection onto $\mathrm{Im}A$. For given $y \in X$, there is a unique $x = x(y)$ in $\mathrm{Im}A$ such that $Ax = p_{\mathrm{Im}A}y$. Forthermore, the mapping $y \mapsto x(y)$ is linear. This mapping is called the pseudo-inverse, or generalized inverse of A.

• **Integration by Parts Formula.** Let Ω be an open and bounded subset of R^d, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial\Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial\Omega$. Let me recall the following fundamental Green's formula, or integration by parts formula: given two functions u, v (with u, v, and all their 1st order partial derivatives belonging to $L^2(\Omega)$, or $u, v \in H^1(\Omega)$), then

$$\int_{\Omega} uv_{x_i} dx = -\int_{\Omega} u_{x_i} v dx + \int_{\partial\Omega} uvn_i dS.$$