Math 273B: Calculus of Variations. **HW #1, due on Monday, Jan 27**

[**1**] Let V be a real vector space and $F : V \to \overline{I\!R}$ be a convex function, thus for every $u, v \in V$, we have

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v),$$

$\forall \lambda \in [0,1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every $u_1, ..., u_n$ points in V and for every family $\lambda_1, ..., \lambda_n$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F(\sum_{i=1}^n \lambda_i u_i) \leq \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \to \overline{I\!R}$ is convex, show that the sections $\{u : \ F(u) \leq a\}$ and $\{u : \ F(u) < a\}$ are convex sets. Show that the converse is not true.

[**2**] The *epigraph* of a function $F : V \to I\!R$ is the set

$$\text{epi}F = \{(u,a) \in V \times I\!R | \ f(u) \leq a\},$$

where V is a normed vector space. Show that the function F is convex if and only if its epigraph is a convex set.

[**3**] Note that a function $F : V \mapsto \overline{I\!R}$, with V a normed vector space, is lower semi-continuous (l.s.c.) on V by the equivalent definition:

$$\forall a \in I\!R : \ \Big\{u \in V | \ F(u) \leq a\Big\} \text{ is closed.}$$

Using this, show that F is l.s.c. iff its epigraph is closed (hint: consider the function on $V \times I\!R$ defined by $f(u,a) = F(u) - a$).

[**4**] Give the definition of an upper semi-continuous (u.s.c.) function. Then, give an example of an u.s.c. function and an example of a function that is not u.s.c.

[**5**] Let $A : I\!R^n \to I\!R^n$ be a linear self-adjoint operator, $b \in I\!R^n$, and consider the quadratic function for $x \in I\!R^n$

$$x \mapsto q(x) := \langle Ax, x \rangle - 2\langle b, x \rangle.$$

Show that the three statements

(i) $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$

(ii) $A \geq O$ and $b \in \text{Im}A$.

(iii) the problem $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$ has a solution

are equivalent. When they hold, characterize the set of minimum points of q, in terms of the pseudo-inverse of A.

Pseudo-Inverse. If A is a symmetric (or self-adjoint) linear operator on X, then $\mathrm{Im}A^{\perp} = \mathrm{Ker}A$. Let $p_{\mathrm{Im}A}$ be the operator of orthogonal projection onto $\mathrm{Im}A$. For given $y \in X$, there is a unique $x = x(y)$ in $\mathrm{Im}A$ such that $Ax = p_{\mathrm{Im}A}y$. Forthermore, the mapping $y \mapsto x(y)$ is linear. This mapping is called the pseudo-inverse, or generalized inverse of A.