Math 273b: Calculus of Variations

Homework #3, due on Wednesday December 30 in class
or in my mailbox during the week of finals

(3 pages)

[1] Consider the 1D length functional minimization problem

$$\min_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \min_u \int_0^1 \sqrt{1 + (u'(x))^2}dx,$$

for twice differentiable functions $u : [0, 1] \to \mathbb{R}$ with boundary conditions $u(0) = 0, \ u(1) = 1$.

(a) Show that the functional $u \mapsto F(u)$ is convex.

(b) Formally compute the Gateaux-differential and then obtain the Euler-Lagrange equation associated with the minimization. Solve the partial differential equation and thus obtain the unique solution of the minimization.

[2] Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear self-adjoint operator, $b \in \mathbb{R}^n$, and consider the quadratic function for $x \in \mathbb{R}^n$

$$x \mapsto q(x) := \langle Ax, x \rangle - 2\langle b, x \rangle.$$

Show that the three statements

(i) $\inf\{q(x) : x \in \mathbb{R}^n\} > -\infty$

(ii) $A \geq O$ and $b \in \text{Im}A$.

(iii) the problem $\inf\{q(x) : x \in \mathbb{R}^n\} > -\infty$ has a solution

are equivalent. When they hold, characterize the set of minimum points of $q$, in terms of the pseudo-inverse of $A$.


(a) Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x))dx,$$

with $u(x_0) = u_0, \ u(x_1) = u_1$ given constants, and $L$ a sufficiently smooth function. Obtain formally the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal $u$.

Hint: Consider smooth test functions $v$, such that $v(x_0) = v(x_1) = 0$. Since $u$ is a minimizer, we must have $F(u) \leq F(u + \varepsilon v)$ for all such sufficiently
smooth functions \( v \) and every real \( \epsilon \). Apply integration by parts to obtain the desired result. You should obtain a second-order differential equation.

(b) Let now \( u(x, t) \) be a smooth solution of the time-dependent partial differential equation (PDE)

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u'}(x, u, u') - L_u(x, u, u'),
\]

with \( u(x, 0) = u_0(x) \) on \((x_0, x_1)\) and \( u(x_0, t) = U_0, u(x_1, t) = U_1 \) for \( t \geq 0 \). Show that the function \( E(t) = F(u(\cdot, t)) \) is decreasing in time, where \( F(u) = \int_{x_0}^{x_1} L(x, u, u') \, dx \).

Optional problems

[1] Consider the minimization problem

\[
\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x)) \, dx,
\]

with \( u(x_0) = u_0, u(x_1) = u_1, u'(x_0) = U_0, u'(x_1) = U_1 \) given, and \( L \) a sufficiently smooth function. As in the previous problem, derive the equation satisfied by a smooth optimal \( u \). Choose test functions \( v \) in \( C^\infty[x_0, x_1] \) that satisfy \( v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0 \). (you should obtain a fourth-order differential equation).

[2] Consider the minimization problem in two dimensions \((x, y)\),

\[
\inf_u E(u) = \int_{\Omega} L(x, y, u, u_x, u_y) \, dxdy, \quad u = g \text{ on } \partial \Omega,
\]

where \( g \) is a given function on the boundary \( \partial \Omega \), with \( \Omega \) a bounded and open region in the plane. Assume that the integrand \( L \) is differentiable.

(i) Show that a sufficiently smooth minimizer \( u \) formally satisfies the Euler-Lagrange equation

\[
\frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P) = 0
\]

on \( \Omega \), where \( P = (x, y, u(x, y), u_x(x, y), u_y(x, y)) \).

(ii) Apply the above result to the case when \( L(x, y, u_x, u_y) = u_x^2 + u_y^2 - 2fu \).

Hint for (i): consider another test function \( v \), such that \( v = 0 \) on \( \partial \Omega \). Since \( u \) is a minimizer, we must have \( E(u) \leq E(u + \epsilon v) \) for all such sufficiently smooth functions \( v \) and all real \( \epsilon \). Apply integration by parts to obtain the desired result. Here, \((u_x, u_y) = \nabla u\).
Notes:

- **Pseudo-Inverse.** If $A$ is a symmetric (or self-adjoint) linear operator on $X$, then $\text{Im} A^+ = \text{Ker} A$. Let $p_{\text{Im} A}$ be the operator of orthogonal projection onto $\text{Im} A$. For given $y \in X$, there is a unique $x = x(y)$ in $\text{Im} A$ such that $Ax = p_{\text{Im} A} y$. Furthermore, the mapping $y \mapsto x(y)$ is linear. This mapping is called the pseudo-inverse, or generalized inverse of $A$.

- **Integration by Parts Formula.** Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial \Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial \Omega$. Let me recall the following fundamental Green’s formula, or integration by parts formula: given two functions $u, v$ (with $u, v$, and all their 1st order partial derivatives belonging to $L^2(\Omega)$, or $u, v \in H^1(\Omega)$), then

$$
\int_{\Omega} uv_x \, dx = - \int_{\Omega} u_x \, v \, dx + \int_{\partial \Omega} uvn_i \, dS.
$$