Math 273b: Calculus of Variations
Homework 2

[**1**] Assume that V and V^* are normed vector spaces in duality. Let $F : V \to \overline{I\!R}$ and let $F^* : V^* \to \overline{I\!R}$ be the polar or conjugate of F. We define F^* by

$$F^*(u^*) = \sup_{u \in V} \Big\{ \langle u^*, u \rangle - F(u) \Big\}.$$

Show
(i) $F^*(0) = -inf_{u \in V} F(u)$.
(ii) $(\lambda F)^*(u^*) = \lambda F^*(\frac{1}{\lambda} u^*)$ for every $\lambda > 0$.

[**2**] Let $V = V^* = I\!R^n$. Let Q be a symmetric positive definite $n \times n$ matrix, $b \in I\!R^n$, and consider $f(x) := \frac{1}{2}\langle x, Qx \rangle + \langle b, x \rangle$, for all $x \in I\!R^n$. Find the polar (or the conjugate) f^* and deduce that, in particular, $\frac{1}{2}\| \cdot \|^2$ is its own polar (or conjugate).

[**3**] Let $F : V \to I\!R$ and F^* its polar. Then $u^* \in \partial F(u)$ if and only if $F(u) + F^*(u^*) = \langle u^*, u \rangle$.

[**4**] Show that the polar F^* is convex.

[**5**] Let $f \in I\!R^{N^2}$ be given, and let $u \in I\!R^{N^2}$ be an unknown minimizer of the functional

$$E(w) = \sum_{i,j=0}^{N-1} |\nabla w_{i,j}|^2 + \lambda \sum_{i,j=0}^{N-1} (w_{i,j} - f_{i,j})^2,$$

for $w \in I\!R^{N^2}$, where

$$\nabla w_{i,j} = \begin{pmatrix} (D_x w)_{i,j} \\ (D_y w)_{i,j} \end{pmatrix} = \begin{pmatrix} w_{i+1,j} - w_{i,j} \\ w_{i,j+1} - w_{i,j} \end{pmatrix},$$

for $(i,j) \in \{0, ..., N-1\}^2$ (we assume that all vectors f, w are periodic outside of their support).

(a) Find the adjoint operators D_x^* and D_y^* of D_x and D_y.

(b) Find a linear operator $B : I\!R^{N^2} \to I\!R^{N^2}$, a $c \in I\!R^{N^2}$, and $C(f)$, independent of w, such that for all $w \in I\!R^{N^2}$,

$$E(w) = \langle Bw, w \rangle + \langle c, w \rangle + C(f).$$

(c) Show that B is self-adjoint.

(d) Find the Gateaux differential of $E(w)$ in the direction v and thus give a necessary (and sufficient) condition for w to be a minimizer, by setting this differential to zero (as the zero functional).