Math 273b: Calculus of Variations
Homework #1

[**1**] Let V be a real vector space and C a convex nonempty subset of V. Suppose that $f : C \mapsto I\!R$ is a convex function. Show that the set of global minimizers of f is a convex set.

[**2**] Let V be a real vector space and $F : V \to \overline{I\!R}$ be a convex function, thus for every $u, v \in V$, we have

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v),$$

$\forall \lambda \in [0,1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every $u_1, ..., u_n$ of points of V and for every family $\lambda_1, ..., \lambda_n$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F(\sum_{i=1}^n \lambda_i u_i) \leq \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \to \overline{I\!R}$ is convex, show that the sections $\{u : \; F(u) \leq a\}$ and $\{u : \; F(u) < a\}$ are convex sets. Show that the converse is not true.

[**3**] The *epigraph* of a function $F : V \to I\!R$ is the set

$$\text{epi}F = \{(u,a) \in V \times I\!R | \; f(u) \leq a\},$$

where V is a normed vector space. Show that the function F is convex if and only if its epigraph is convex.

[**4**] Note that a function $F : V \mapsto \overline{I\!R}$, with V a normed vector space, is lower semi-continuous (l.s.c.) on V by the equivalent definition:

$$\forall a \in I\!R : \; \Big\{u \in V | \; F(u) \leq a\Big\} \text{ is closed.}$$

Using this (or another method), show that F is l.s.c. iff its epigraph is closed (hint: consider the function on $V \times I\!R$ defined by $f(u,a) = F(u) - a$).

[**5**] Let $F : I\!R^n \to I\!R$ be continuous and coercive with respect to the Euclidean norm $\|x\|$, for $x \in I\!R^n$. Show: then there is $x_0 \in I\!R^n$ such that $F(x_0) = \inf_{x \in I\!R^n} F(x)$. If in addition F is strictly convex, then the minimizer x_0 is unique (give a simplified version of the proof given in class for the more general case).