Math 273: Homework #3, due on Friday, November 14

[1] (This problem is related with [2] from Hw #2)

Let $u(x, y, t)$ be a smooth solution of the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P),$$

with $u(x, y, 0) = u_0(x, y)$ in Ω and $u(x, y, t) = g(x, y)$ for $(x, y) \in \partial\Omega$ and $t \geq 0$ (recall that P is a notation for $(x, y, u(x, y), u_x(x, y), u_y(x, y))$).

Show that the function $E(t) = F(u(\cdot, \cdot, t))$ is decreasing in time, where $F(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy$.

[2] Let Ω be an open and bounded domain in $\mathbb{R}^2$, with ufficiently smooth boundary $\partial\Omega$. Consider the minimization problem in two dimensions

$$\inf_u F(u) = \int_\Omega (Ku - u_0)^2 dxdy + \alpha \int_\Omega f(\nabla u) dxdy,$$

with $u_0 \in L^2(\Omega)$ a given square-integrable function, and f a smooth function on $\mathbb{R}^2$ with real values. Here $K : L^2(\Omega) \to L^2(\Omega)$ is a linear and continuous operator, and its adjoint is K^* (thus K^* has the property $\int_\Omega (Ku) v dxdy = \int_\Omega u(K^* v) dxdy$). Obtain, as before, the Euler-Lagrange equation associated with the minimization problem that is formally satisfied by a sufficiently smooth optimal u. No explicit boundary conditions are imposed, thus you have to deduce implicit (or natural) boundary conditions on $\partial\Omega$.

[3] Apply the gradient descent method described in class to the two-dimensional diffusion problem

$$F(u) = \sum_{1 \leq i,j \leq n} \Big[(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \lambda (u_{i,j} - f_{i,j})^2 \Big],$$

where $f_{i,j}$ is given for $0 \leq i, j \leq n+1$, and with boundary conditions $u_{i,j} = f_{i,j}$ if $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$ (chosen for simplicity). Here $\lambda > 0$ is a tunning parameter. Choose a function f of your choice (for example an image). If you do not have one, you can create a synthetic image. Test various values of the parameter λ and observe the properties of your implementation. Give your choice of the stopping criterion and also plot the value of the objective function versus iterations. Plot the data f, your starting point and your final result, as 2D images.

[4] Recall the BFGS update formula for the Hessian approximation:

$$B_{k+1} = B_k - \frac{B_k s_k s_k^t B_k}{s_k^t B_k s_k} + \frac{y_k y_k^t}{y_k^t s_k}$$

(where B_k is symmetric and positive definite), and the formula to directly update the inverse of Hessian approximation:

$$H_{k+1} = (I - \rho_k s_k y_k^t) H_k (I - \rho_k y_k s_k^t) + \rho_k s_k s_k^t$$

(where H_k is symmetric and positive definite, as inverse of B_k, and $\rho_k = \frac{1}{y_k^t s_k}$).

Using the following Sherman-Morrison-Woodbury formula, show that H_{k+1} is the inverse of B_{k+1}.

(SWM) If A is an $n \times n$ nonsingular matrix, and a, b vectors in $\mathbb{R}^n$, let $\overline{A} = A + ab^t$, then the following (SMW) formula holds:

$$\overline{A}^{-1} = A^{-1} - \frac{A^{-1} a b^t A^{-1}}{1 + b^t A^{-1} a}.$$

[5] Consider the constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{subject to } Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $m \leq n$, $b \in \mathbb{R}^m$. Tranform the problem into an unconstrained minimization problem by one of the methods discussed in class.

[6] Verify that the KKT conditions (1st order optimality conditions) for the bound-constrained problem

$$\min_{x \in \mathbb{R}^n} \phi(x), \text{ subject to } l \leq x \leq u,$$

are equivalent to the compactly stated condition $P_{[l,u]} \nabla \phi(x) = 0$, where the projection operator $P_{[l,u]}$ of a vector $g \in \mathbb{R}^n$ onto the rectangular box $[l, u]$ is defined by

$$(P_{[l,u]} g)_i = \begin{cases} min(0, g_i), & \text{if } x_i = l_i, \\ g_i, & \text{if } x_i \in (l_i, u_i), \text{ for all } i = 1, 2, ..., n \\ max(0, g_i), & \text{if } x_i = u_i. \end{cases}$$