[1] Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the function
$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$ Show that $x^* = (1, 1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

[2] Let $a$ be a given $n$-vector, and $A$ be a given $n \times n$ symmetric matrix. Compute the gradient and Hessian of $f_1(x) = a^T x$ and $f_2(x) = x^T Ax$.

[3] Suppose that $f$ is a convex function. Show that the set of global minimizers of $f$ is a convex set.

[4] Suppose that $\hat{f}(z) = f(x)$, where $x = Sz + s$ for some $S \in R^{n \times n}$ and $s \in R^n$. Show that
$$\nabla \hat{f}(z) = S^T \nabla f(x), \quad \nabla^2 \hat{f}(z) = S^T \nabla^2 f(x) S.$$ [5] Computation of the Euler-Lagrange equation in the continuous case.
(a) Consider the minimization problem
$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x))dx,$$ with $u(x_0) = u_0$, $u(x_1) = u_1$ given constants, and $L$ a sufficiently smooth function. Obtain formally the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal $u$.

Hint: Consider test functions $v$, such that $v(x_0) = v(x_1) = 0$. Since $u$ is a minimizer, we must have $F(u) \leq F(u + \epsilon v)$ for all such sufficiently smooth functions $v$ and every real $\epsilon$. Apply integration by parts to obtain the desired result. You should obtain a second-order differential equation.

(b) Let now $u(x, t)$ be a smooth solution of the time-dependent partial differential equation (PDE)
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_u(x, u, u') - L_u(x, u, u'),$$ with $u(x, 0) = u_0(x)$ on $(x_0, x_1)$ and $u(x_0, t) = U_0$, $u(x_1, t) = U_1$ for $t \geq 0$. Show that the function $E(t) = F(u(\cdot, t))$ is decreasing in time, where $F(u) = \int_{x_0}^{x_1} L(x, u, u')dx$. 

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Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial \Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial \Omega$. Recall the following fundamental Green’s formula, or integration by parts formula: given two functions $u, v$ (with $u, v, \text{ and all their } 1\text{st order partial derivatives belonging to } L^2(\Omega), \text{ or } u, v \in H^1(\Omega)$), then

$$\int_{\Omega} u v_x dx = - \int_{\Omega} u_x v dx + \int_{\partial \Omega} u v n_i dS.$$