Math 273 **Homework #4** Due on Friday, March 22nd
(you can leave your homework in my mailbox, under the door at my office MS 7620-D, or with Babette Dalton at MS 7619 before 3pm every day, or you can send it by email).

[1] Let V be a real vector space and $F : V \to \overline{\mathbb{R}}$ be a convex function, thus for every $u, v \in V$, we have $F(\lambda u + (1-\lambda)v) \le \lambda F(u) + (1-\lambda)F(v)$, $\forall \lambda \in [0,1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every $u_1, ..., u_n$ of points of V and for every family $\lambda_1, ..., \lambda_n$, $\lambda_i \ge 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F(\sum_{i=1}^n \lambda_i u_i) \le \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \to \overline{\mathbb{R}}$ is convex, show that the sets $\{u : \ F(u) \le a\}$ and $\{u : \ F(u) < a\}$ are convex sets. Show that the converse is not true.

[2] The *epigraph* of a function $F : V \to \mathbb{R}$ is the set

$$\text{epi}F = \{(u,a) \in V \times \mathbb{R} |\ f(u) \le a\},$$

where V is a Banach space. Show that the function F is convex if and only if its epigraph is convex.

[3] Assume that V and V^* are normed vector spaces in duality. Let $F : V \to \overline{\mathbb{R}}$ and let $F^* : V^* \to \overline{\mathbb{R}}$ be the polar or conjugate of F. We define F^* by

$$F^*(u^*) = \sup_{u \in V} \Big\{ \langle u^*, u \rangle - F(u) \Big\},$$

where $\langle u^*, u \rangle = u^*(u)$.

Show

(i) $F^*(0) = -inf_{u \in V} F(u)$.

(ii) $(\lambda F)^*(u^*) = \lambda F^*(\frac{1}{\lambda} u^*)$ for every $\lambda > 0$.

[4] Let $V = V^* = \mathbb{R}^n$. Let Q be a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$, and consider $f(x) := \frac{1}{2}\langle x, Qx \rangle + \langle b, x \rangle$, for all $x \in \mathbb{R}^n$. Find the polar (or the conjugate) f^* and deduce that, in particular, $\frac{1}{2}\| \cdot \|^2$ is its own polar (or conjugate).

[5] Let $F : V \to \mathbb{R}$ and F^* its polar. Then $u^* \in \partial F(u)$ if and only if $F(u) + F^*(u^*) = \langle u^*, u \rangle$.

[6] Show that the polar F^* is convex.

[7] Let $f \in \mathbb{R}^{N^2}$ be given, and let $u \in \mathbb{R}^{N^2}$ be an unknown minimizer of the functional (already seen before)

$$E(w) = \sum_{i,j=0}^{N-1} |\nabla w_{i,j}|^2 + \lambda \sum_{i,j=0}^{N-1} (w_{i,j} - f_{i,j})^2,$$

for $w \in \mathbb{R}^{N^2}$, where

$$\nabla w_{i,j} = \begin{pmatrix} (D_x w)_{i,j} \\ (D_y w)_{i,j} \end{pmatrix} = \begin{pmatrix} w_{i+1,j} - w_{i,j} \\ w_{i,j+1} - w_{i,j} \end{pmatrix},$$

for $(i,j) \in \{0, ..., N-1\}^2$ (we assume that all vectors f, u, w are periodic outside of their support).

(a) Find the adjoint operators D_x^* and D_y^* of D_x and D_y.

(b) Find a linear operator $B : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$, a $c \in \mathbb{R}^{N^2}$, and $C(f)$, independent of w, such that for all $w \in \mathbb{R}^{N^2}$,

$$E(w) = \langle Bw, w\rangle + \langle c, w\rangle + C(f).$$

(c) Show that B is self-adjoint.

(d) Find the Gateaux differential of $E(w)$ in the direction v and thus give a necessary (and sufficient) condition for u to be a minimizer, by setting this differential to zero (as the zero functional).

Optional Problems

[1] Note that a function $F : V \mapsto \overline{\mathbb{R}}$, with V a normed vector space, is lower semi-continuous (l.s.c.) on V by the equivalent definition:

$$\forall a \in \mathbb{R} : \quad \Big\{ u \in V |\ F(u) \le a \Big\} \text{ is closed.}$$

Using this, show that F is l.s.c. iff its epigraph is closed (hint: consider the function on $V \times \mathbb{R}$ defined by $f(u, a) = F(u) - a$).

[2] Infimal convolution. Let $f_1, \ . \ \ . \ \ . \ , f_m$ be convex functions on $\mathbb{R}^n$. Their infimal convolution, denoted $g = f_1 \circ ... \circ f_m$ (several other notations are also used), is defined as $g(x) = \inf f_1(x_1) + ... + f_m(x_m)|x_1 + ... + x_m = x$, with the natural domain (i.e., defined by $g(x) < \infty$).

(The name "convolution" presumably comes from the observation that if we replace the sum above with the product, and the infimum above with integration, then we obtain the normal convolution.)

(a) Show that g is convex.

(b) Show that $g^* = f_1^* + ... + f_m^*$.

[3] *Approximate total variation denoising.* Let $y \in \mathbb{R}^n$ be a given (corrupted, noisy) vector, $x \in \mathbb{R}^n$ is the denoised vector to be computed by minimizing the 1D-discrete TV function with data fidelity term:

$$f(x) = \|x - y\|_2^2 + \lambda \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

We can make the second term twice differentiable using regularization:

$$TV(x) = \sum_{i=1}^{n-1} |x_{i+1} - x_i| \approx \sum_{i=1}^{n-1} \Big(\sqrt{\epsilon^2 + (x_{i+1} - x_i)^2} - \epsilon \Big),$$

where $\epsilon > 0$ is a small parameter. Apply gradient descent method and Newton's method for the minimization using the approximate formula for TV (simlar problem with the previous length minimization). Define a noisy vector y as data.