Math 273 **Homework #3** Due on Monday, March 11

[1] Show that $(0,-1)^T$ is a local minimizer for the problem
Minimize $f(x) = 2x_1^2 + x_2$ subject to
$x_2 \geq x_1^2 - 1$
$x_1 \geq x_2$.

[2] Verify that the KKT conditions (1st order optimality conditions) for the bound-constrained problem

$$\min_{x \in \mathbb{R}^n} \phi(x), \text{ subject to } l \leq x \leq u,$$

are equivalent to the compactly stated condition $P_{[l,u]} \nabla \phi(x) = 0$, where the projection operator $P_{[l,u]}$ of a vector $g \in \mathbb{R}^n$ onto the rectangular box $[l,u]$ is defined by

$$(P_{[l,u]} g)_i = \begin{cases} min(0, g_i), & \text{if } x_i = l_i, \\ g_i, & \text{if } x_i \in (l_i, u_i), \text{ for all } i = 1, 2, ..., n \\ max(0, g_i), & \text{if } x_i = u_i. \end{cases}$$

[3] Repeat problem [1] from Hw #2 using now Newton's method, and compare the two methods. Give details about your implementation (computation of gradient, of Hessian, of its inverse, selection of α, stopping criterion), and include your code.

[4] Consider the minimization problem in two dimensions $(x,y) \in \Omega$,

$$\inf_u \Big\{ F(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy, \quad u = g \text{ on } \partial\Omega \Big\},$$

where g is a given function on the boundary $\partial\Omega$, with Ω an open and bounded region in the plane. Assume that the integrand L is differentiable.

(i) Show that a sufficiently smooth minimizer u formally satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial x}\Big(L_{u_x}(P)\Big) + \frac{\partial}{\partial y}\Big(L_{u_y}(P)\Big) - L_u(P) = 0$$

on Ω, where $P = (x, y, u(x,y), u_x(x,y), u_y(x,y))$.

(iii) Let $u(x,y,t)$ be a smooth solution of the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(L_{u_x}(P)\Big) + \frac{\partial}{\partial y}\Big(L_{u_y}(P)\Big) - L_u(P),$$

with $u(x,y,0) = u_0(x,y)$ in Ω and $u(x,y,t) = g(x,y)$ for $(x,y) \in \partial\Omega$ and $t \geq 0$.

Show that the function $E(t) = F(u(\cdot,\cdot,t))$ is decreasing in time.

Hint for (i): consider another test function v, such that $v = 0$ on $\partial\Omega$. Since u is a minimizer, we must have $F(u) \leq F(u + \varepsilon v)$ for all such sufficiently smooth functions v and all real ϵ. Apply integration by parts to obtain the desired result in (i).

Here, $(u_x, u_y) = \nabla u = \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)$.

[5] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$, $u'(x_0) = U_0$, $u'(x_1) = U_1$ given, and L a sufficiently smooth function. As in the previous problem, derive the equation satisfied by a smooth optimal u. Choose test functions v in $C^\infty[x_0, x_1]$ that satisfy $v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0$. (you should obtain a fourth-order differential equation).

Notes

Let Ω be an open and bounded subset of $\mathbb{R}^d$, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial\Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial\Omega$. Recall the following fundamental Green's formula, or integration by parts formula: given two functions u, v (with u, v, and all their 1st order partial derivatives belonging to $L^2(\Omega)$, or $u, v \in H^1(\Omega)$), then

$$\int_\Omega u v_{x_i} dx = -\int_\Omega u_{x_i} v dx + \int_{\partial\Omega} u v n_i dS.$$