Suppose that \( f \) is a convex function. Show that the set of global minimizers of \( f \) is a convex set.

Consider the 1D length functional minimization problem

\[
\min_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \min_u \int_0^1 \sqrt{1 + (u'(x))^2} dx,
\]

over functions \( u : [0, 1] \to \mathbb{R} \) with boundary conditions \( u(0) = 0, u(1) = 1 \).

(a) Find the exact solution of the problem. Justify your answer.

(b) Show that the functional \( u \mapsto F(u) \) is convex.

(c) Consider a discrete version of the problem: let

\[
x_0 = 0 < x_1 < ... < x_n < x_{n+1} = 1
\]

be equidistant points, with \( x_{i+1} - x_i = h \). For \( \vec{u} = (u_1, ..., u_n) \), consider

\[
f(\vec{u}) = \sum_{i=0}^{n} \sqrt{1 + \left( \frac{u_{i+1} - u_i}{h} \right)^2},
\]

with the additional condition that \( u_0 = 0 \) and \( u_{n+1} = 1 \).

Choose an appropriate discretization integer \( n \). Then numerically and experimentally analyze the behavior of the gradient descent method with backtracking line search. Choose the initial starting point \( u^0 \) as a curve joining the points \((0, 0)\) and \((1, 1)\). Record the number of iterations and plot the error \( u_k - u^* \), where \( u^* \) is the exact minimizer. You could also plot the curve given by \( \vec{u}_k \) at some iterations.

(d) Repeat question (c), using now Newton’s method.

(e) Discuss the results obtained in (c) and (d).

Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be a (linear) self-adjoint operator, \( b \in \mathbb{R}^n \), and consider the quadratic function for \( x \in \mathbb{R}^n \)

\[
x \mapsto q(x) := \langle Ax, x \rangle - 2\langle b, x \rangle.
\]

Show that the three statements

(i) \( \inf \{ q(x) : x \in \mathbb{R}^n \} > -\infty \)

(ii) \( A \geq O \) and \( b \in \text{Im}A \).

(iii) the problem \( \inf \{ q(x) : x \in \mathbb{R}^n \} > -\infty \) has a solution

are equivalent. When they hold, characterize the set of minimum points of \( q \), in terms of the pseudo-inverse of \( A \).