Math 273: Homework #1, due on Wednesday, January 30

[**1**] Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

[**2**] Consider the 1D length functional minimization problem

$$\text{Min}_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \text{Min}_u \int_0^1 \sqrt{1+(u'(x))^2}dx,$$

over functions $u : [0,1] \to I\!R$ with boundary conditions $u(0) = 0$, $u(1) = 1$.

(a) Find the exact solution of the problem. Justify your answer.

(b) Show that the functional $u \mapsto F(u)$ is convex.

(c) Consider a discrete version of the problem: let

$$x_0 = 0 < x_1 < ... < x_n < x_{n+1} = 1$$

be equidistant points, with $x_{i+1} - x_i = h$. For $\vec{u} = (u_1, ..., u_n)$, consider $f(\vec{u}) = \sum_{i=0}^n \sqrt{1 + \left(\frac{u_{i+1}-u_i}{h}\right)^2}$, with the additional condition that $u_0 = 0$ and $u_{n+1} = 1$.

Choose an appropriate discretization integer n. Then numerically and experimentally analyze the behavior of the gradient descent metho d with backtracking line search. Choose the initial starting point u^0 as a curve joining the points $(0,0)$ and $(1,1)$. Record the number of iterations and plot the error $u^k - u^*$, where u^* is the exact minimizer. You could also plot the curve given by $\vec{u}^k$ at some iterations.

(d) Repeat question (c), using now Newton's method.

(e) Discuss the results obtained in (c) and (d).

[**3**] Let $A : I\!R^n \to I\!R^n$ be a (linear) self-adjoint operator, $b \in I\!R^n$, and consider the quadratic functi on for $x \in I\!R^n$

$$x \mapsto q(x) := \langle Ax, x\rangle - 2\langle b, x\rangle.$$

Show that the three statements

(i) $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$

(ii) $A \geq O$ and $b \in \text{Im}A$.

(iii) the problem $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$ has a solution

are equivalent. When they hold, characterize the set of minimum points of q, in terms of the pseudo-inverse of A.