Notes on Stable and Normal Problems

Assume that $\Phi : V \times Y \to \overline{\mathbb{R}}$ is convex and l.s.c. such that Φ does not take the value $-\infty$ and is not identically $+\infty$.

We define $h : Y \to \mathbb{R}$ by $h(p) = \inf(\mathcal{P}_p) = \inf_{v\in V} \Phi(u, p)$.

Def. 1. Problem $\mathcal{P}$ is said to be *normal* if $h(0)$ is finite and h is l.s.c.

Def. 2. Problem $\mathcal{P}$ is said to be *stable* if $h(0)$ is finite and h is subdifferentiable at 0.

Proposition 1. The following three conditions are equivalent:
(i) $\mathcal{P}$ is normal
(ii) $\mathcal{P}^*$ is normal
(iii) $\inf \mathcal{P} = \sup \mathcal{P}^*$ and this number is finite

Proposition 2. The following two conditions are equivalent:
(i) $\mathcal{P}$ is stable
(ii) $\mathcal{P}$ is normal and $\mathcal{P}^*$ has at least one solution.

Corollary 1. The following three conditions are equivalent:
(i) $\mathcal{P}$ and $\mathcal{P}^*$ are normal and have some solutions
(ii) $\mathcal{P}$ and $\mathcal{P}^*$ are stable
(iii) $\mathcal{P}$ is stable and has some solutions.

Proposition 3. A stability criterion.
Assume that Φ is convex, that $\inf \mathcal{P}$ is finite and that
(1) $\Big[$There is $u_0 \in V$ s.t. $p \mapsto \Phi(u_0, p)$ is finite and continuous at 0 ($\in Y$). $\Big]$
Then problem $\mathcal{P}$ is stable.

Theorem Assume that V is a reflexive Banach space, (1) satisfied, Φ as above, and $\lim \Phi(u, 0) = +\infty$ if $\|u\|_V \to \infty$. Then $\mathcal{P}$ and $\mathcal{P}^*$ each have at least one solution,

$$\inf \mathcal{P} = \sup \mathcal{P}^*$$

and the extremality relation is satisfied:

$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}) = 0 \Leftrightarrow (0, \bar{p}^*) \in \partial\Phi(\bar{u}, 0).$$

Proof. By the assumptions, from the general existence theorem, we have existence of a solution of problem $\mathcal{P}$. Proposition 3 implies that $\mathcal{P}$ is stable. By the above results, this implies that $\mathcal{P}^*$ also has solutions, and that inf=sup=finite. The extremality relation follows from the Thm. done in class.