

Math 273**Homework #4, due on Monday, November 22nd**

[1] Consider the quadratic program

$$\min_x q(x) = \frac{1}{2}x^T Gx + x^T d, \text{ subject to } Ax = b,$$

where $G \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $A \in \mathbb{R}^{m \times n}$. Assume that A has full row rank m .

(a) Express the first order necessary conditions for x_* to be a solution in the form of a linear matrix equation in the unknown $(x_* \ \lambda^*)^T$.

(b) Express in (a) x_* by $x + p$, with x some fixed feasible estimate and unknown $p \in \text{Null}(A)$. Re-write the matrix equation now in the unknown $(-p \ \lambda^*)^T$.

(c) Assume in addition that the reduced-Hessian $Z^T GZ$ is positive definite. Show that the coefficient matrix in (b) is non-singular, thus there is a unique vector pair (x_*, λ^*) satisfying the matrix equation in (a).

[2] Show that $(0, -1)^T$ is a local minimizer for the problem

$$\text{Minimize } f(x) = 2x_1^2 + x_2 \text{ subject to}$$

$$x_2 \geq x_1^2 - 1$$

$$x_1 \geq x_2.$$

[3] The problem of finding the shortest distance from a point x_0 to the hyperplane $\{x : Ax = b\}$ where A has full row rank can be formulated as the quadratic program

$$\min \frac{1}{2}(x - x_0)^T(x - x_0), \quad \text{s.t. } Ax = b.$$

(i) Show that the optimal multiplier is $\lambda^* = (AA^T)^{-1}(b - Ax_0)$, and that the solution is $x_* = x_0 + A^T(AA^T)^{-1}(b - Ax_0)$.

(ii) Show that in the special case where A is a row vector, the shortest distance from x_0 to the solution set of $Ax = b$ is $\frac{|b - Ax_0|}{\|A\|}$.

[4] Repeat problem [3], Hw #3 using now Newton's method, and compare the two methods. Give details about your implementation (computation of gradient, of Hessian, of inverse, about α , about your stopping criteria, etc), and include your code.

[5] Let V be a real vector space and $F : V \rightarrow \overline{\mathbb{R}}$ be a convex function, thus for every $u, v \in \mathcal{V}$, we have $F(\lambda u + (1 - \lambda)v) \leq \lambda F(u) + (1 - \lambda)F(v)$, $\forall \lambda \in [0, 1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every u_1, \dots, u_n of points of V and for every family $\lambda_1, \dots, \lambda_n$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F\left(\sum_{i=1}^n \lambda_i u_i\right) \leq \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \rightarrow \overline{\mathbb{R}}$ is convex, show that the sections $\{u : F(u) \leq a\}$ and $\{u : F(u) < a\}$ are convex sets. Show that the converse is not true.

[6] The *epigraph* of a function $F : V \rightarrow \mathbb{R}$ is the set

$$\text{epi}F = \{(u, a) \in V \times \mathbb{R} \mid f(u) \leq a\},$$

where V is a Banach space. Show that the function F is convex if and only if its epigraph is convex.