Math 273: **Homework #2, due on Monday, October 25**

[**1**] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x))dx,$$

with $u(x_0) = u_0$, $u(x_1) = u_1$, $u'(x_0) = U_0$, $u'(x_1) = U_1$ given, and L is a sufficiently smooth function. Obtain the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal u. Choose test functions v in $C^\infty[x_0, x_1]$ that satisfy $v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0$, and proceed as in HW1, problem [5] (you should obtain a fourth-order differential equation).

[**2**] Consider the 1D length functional minimization problem

$$\text{Min}_u F(u) = \int_0^1 L(u'(x))dx, \text{ or } \text{Min}_u \int_0^1 \sqrt{1 + (u'(x))^2}dx,$$

over functions $u : [0, 1] \to I\!R$ with boundary conditions $u(0) = 0$, $u(1) = 1$.

(a) Find the exact solution of the problem.

(b) Show that the functional $u \mapsto F(u)$ is convex.

(c) Consider a discrete version of the problem: let

$$x_0 = 0 < x_1 < ... < x_n < x_{n+1} = 1$$

be equidistant points, with $x_{i+1} - x_i = h$. For $\vec{u} = (u_1, ..., u_n)$, consider $f(\vec{u}) = \sum_{i=0}^n \sqrt{1 + \left(\frac{u_{i+1} - u_i}{h}\right)^2}$, with the additional condition that $u_0 = 0$ and $u_{n+1} = 1$.

Choose an appropriate discretization integer n. Then numerically and experimentally analyze the behavior of the gradient descent method with backtracking line search. Choose the initial starting point u^0 as a curve joining the points $(0, 0)$ and $(1, 1)$. Record the number of iterations and plot the error $u^k - u^*$, where u^* is the exact minimizer. You could also plot the curve given by $\vec{u}^k$ at some iterations.

(d) Repeat question (c), using now Newton's method.

(e) Discuss the results obtained in (c) and (d).

[**3**] Let $A : I\!R^n \to I\!R^n$ be a (linear) self-adjoint operator, $b \in I\!R^n$, and consider the quadratic function for $x \in I\!R^n$

$$x \mapsto q(x) := \langle Ax, x\rangle - 2\langle b, x\rangle.$$

Show that the three statements

(i) $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$

(ii) $A \geq O$ and $b \in \text{Im}A$.

(iii) the problem $\inf\{q(x) : \ x \in I\!R^n\} > -\infty$ has a solution

are equivalent. When they hold, characterize the set of minimum points of q, in terms of the pseudo-inverse of A.

[4] Recall the BFGS update formula for the Hessian approximation:

$$B_{k+1} = B_k - \frac{B_k s_k s_k^t B_k}{s_k^t B_k s_k} + \frac{y_k y_k^t}{y_k^t s_k}$$

(where B_k is symmetric and positive definite), and the formula to directly update the inverse of Hessian approximation:

$$H_{k+1} = (I - \rho_k s_k y_k^t) H_k (I - \rho_k y_k s_k^t) + \rho_k s_k s_k^t$$

(where H_k is symmetric and positive definite, as inverse of B_k, and $\rho_k = \frac{1}{y_k^t s_k}$).

Using the following Sherman-Morrison-Woodbury formula, show that H_{k+1} is the inverse of B_{k+1}.

If A is an $n \times n$ nonsingular matrix, and a, b vectors in $I\!R^n$, let $\overline{A} = A + ab^t$. Then the following (SMW) formula holds:

$$\text{(SMW)} \qquad \overline{A}^{-1} = A^{-1} - \frac{A^{-1} a b^t A^{-1}}{1 + b^t A^{-1} a}.$$

Notes:

• If A is a symmetric (or self-adjoint) linear operator on X, then $\mathrm{Im}A^\perp = \mathrm{Ker}A$. Let $p_{\mathrm{Im}A}$ be the operator of orthogonal projection onto $\mathrm{Im}A$. For given $y \in X$, there is a unique $x = x(y)$ in $\mathrm{Im}A$ such that $Ax = p_{\mathrm{Im}A} y$. Forthermore, the mapping $y \mapsto x(y)$ is linear. This mapping is called the pseudo-inverse, or generalized inverse of A.

• Let Ω be an open and bounded subset of R^d, with Lipschitz-continuous (or sufficiently smooth) boundary $\partial\Omega$. Let $\vec{n} = (n_1, n_2, ..., n_d)$ be the exterior unit normal to $\partial\Omega$. Recall the following fundamental Green's formula, or integration by parts formula: given two functions u, v (with u, v, and all their 1st order partial derivatives belonging to $L^2(\Omega)$, or $u, v \in H^1(\Omega)$), then

$$\int_\Omega u v_{x_i} dx = -\int_\Omega u_{x_i} v dx + \int_{\partial\Omega} u v n_i dS.$$