Math 273: **Homework #1, due on Monday, October 11**

1. Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Show that $x^* = (1,1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

2. Let a be a given n-vector, and A be a given $n \times n$ symmetric matrix. Compute the gradient and Hessian of $f_1(x) = a^T x$ and $f_2(x) = x^T A x$.

3. Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

4. Suppose that $\hat{f}(z) = f(x)$, where $x = Sz + s$ for some $S \in R^{n \times n}$ and $s \in R^n$. Show that

$$\nabla \hat{f}(z) = S^T \nabla f(x), \qquad \nabla^2 \hat{f}(z) = S^T \nabla^2 f(x) S.$$

5. *Computation of the Euler-Lagrange equation in the continuous case*
Consider the minimization problem in two dimensions (x, y),

$$\inf_u E(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy, \quad u = g \text{ on } \partial\Omega,$$

where g is a given function on the boundary $\partial\Omega$, with Ω an bounded and bounded region in the plane. Assume that the integrand L is differentiable.

(i) Show that a sufficiently smooth minimizer u formally satisfies the Euler-Lagrange equation

$$\frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P) = 0$$

on Ω, where $P = (x, y, u(x, y), u_x(x, y), u_y(x, y))$.

(ii) Apply the above result to the case when $L(x, y, u_x, u_y) = u_x^2 + u_y^2 - 2fu$.

Hint: consider another test function v, such that $v = 0$ on $\partial\Omega$. Since u is a minimizer, we must have $E(u) \leq E(u + \varepsilon v)$ for all such sufficiently smooth functions v and all real ϵ. Apply integration by parts to obtain the desired result. Here, $(u_x, u_y) = \nabla u$.