Summary of necessary and sufficient conditions for local minimizers

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Unconstrained problem $\min\limits_{x \in R^n} f(x)$

1st-order necessary conditions If x_* is a local minimizer of f and f is continuously differentiable in an open neighborhood of x_*, then
- $\nabla f(x_*) = \vec{0}$.

2nd-order necessary conditions If x_* is a local minimizer of f and $\nabla^2 f$ is continuous in an open neighborhood of x_*, then
- $\nabla f(x_*) = \vec{0}$
- $\nabla^2 f(x_*)$ is positive semi-definite.

2nd-order sufficient conditions Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x_*. If the following two conditions are satisfied, then x_* is a strict local minimizer of f:
- $\nabla f(x_*) = \vec{0}$
- $\nabla^2 f(x_*)$ is positive definite.

Equality constrained problem $\min\limits_{x \in R^n} f(x)$, subject to $c_i(x) = 0$, $i = 1, ..., m$, $m \leq n$

Necessary conditions Assume f and c_i are twice continuously differentiable in an open neighborhood of x_*, that $\nabla c_i(x_*)$ are linearly independent vectors, and that x_* is a local minimizer. Let $A(x_*) \in R^{m \times n}$ be the Jacobian of all constraints at x_* (of full row rank), and $Z(x_*) \in R^{n \times (m-n)}$ be a basis matrix for the null space of $A(x_*)$. Then there is a Lagrange multiplier $\lambda^* \in R^m$ such that
- $c_i(x_*) = 0$ for $i = 1, ..., m$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = \vec{0} \Leftrightarrow \nabla f(x_*) = A(x_*)^T \lambda_*$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda^*) Z(x_*)$ is positive semi-definite

Sufficient conditions Assume f and c_i are twice continuously differentiable in an open neighborhood of x_*, and that $\nabla c_i(x_*)$ are linearly independent vectors at x_*. Let $A(x_*) \in R^{m \times n}$ be the Jacobian of all constraints at x_* (of full row rank), and $Z(x_*) \in R^{n \times (m-n)}$ be a basis matrix for the null space of $A(x_*)$. If there is a $\lambda^* \in R^m$ such that the following conditions are satisfied at (x_*, λ^*), then x_* is a strict local minimizer:
- $c_i(x_*) = 0$ for $i = 1, ..., m$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = \vec{0} \Leftrightarrow \nabla f(x_*) = A(x_*)^T \lambda_*$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda^*) Z(x_*)$ is positive definite

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Inequality constrained problem $\min\limits_{x \in R^n} f(x)$, subject to $c_i(x) \geq 0$, $i = 1, ..., m$, $m \leq n$

Necessary conditions Let x_* be a local minimizer, such that the linear independence of $\nabla c_i(x_*)$ for all active constraints holds. Let $Z(x_*)$ be a basis matrix for the null space of the Jacobian of the active constraints at x_*. Then there is a Lagrange multiplier $\lambda^* \in R^m$ such that
- $c_i(x_*) \geq 0$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = 0$
- $\lambda_i^* \geq 0$
- $\lambda^{*T} c(x_*) = 0$
- $Z(x_*)^T \nabla^2_{xx} \mathcal{L}(x_*, \lambda_*) Z(x_*)$ is positive semi-definite.

<u>Sufficient conditions</u> Let x_* be such that the linear independence of $\nabla c_i(x_*)$ for all active constraints holds. Let $Z(x_*)$ be a basis matrix for the null space of the Jacobian of the active constraints at x_*. If there is a Lagrange multiplier $\lambda^* \in R^m$ such that the following conditions are satisfied at (x_*, λ^*), then x_* is a strict local minimizer:

- $c_i(x_*) \geq 0$
- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0 \Leftrightarrow Z(x_*)^T \nabla f(x_*) = 0$
- $\lambda_i^* \geq 0$, and $\lambda_i^* > 0$ if $c_i(x_*) = 0$
- $\lambda^{*T} c(x_*) = 0$
- $Z(x_*)^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda_*) Z(x_*)$ is positive definite.

Equality and inequality constrained problem $\quad \min\limits_{x \in R^n} f(x)$, subject to $\begin{cases} c_i(x) = 0, \; i \in \mathcal{E} \\ c_i(x) \geq 0, \; i \in \mathcal{I} \end{cases}$

<u>1st-order necessary conditions</u> Let $\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} : \; c_i(x) = 0\}$ be the set of all active constraints at a point x. Assume that at a point x_*, the active constraints gradients $\nabla c_i(x_*)$, $i \in \mathcal{A}(x_*)$ are linearly independent. Assume that x_* is a local minimizer. Then there is a Lagrange multiplier $\lambda^* = (\lambda^*)_i$, $i \in \mathcal{E} \cup \mathcal{I}$, such that the following conditions are satisfied at (x_*, λ^*):

- $\nabla_x \mathcal{L}(x_*, \lambda^*) = 0$
- $c_i(x_*) = 0$, for all $i \in \mathcal{E}$
- $c_i(x_*) \geq 0$, for all $i \in \mathcal{I}$
- $\lambda_i^* \geq 0$, for all $i \in \mathcal{I}$
- $\lambda_i^* c_i(x_*) = 0$, for all $i \in \mathcal{E} \cup \mathcal{I}$

(these conditions are called the *KKT conditions* from Karush-Kunn-Tucker conditions).

Note that for inactive constraints $c_i(x_*) > 0$ we have $\lambda_i^* = 0$, thus the condition on the Lagrangian function becomes

$$ 0 = \nabla_x \mathcal{L}(x_*, \lambda^*) = \nabla f(x_*) - \sum_{i \in \mathcal{A}(x_*)} \lambda_i^* \nabla c_i(x_*). $$

<u>2nd-order necessary conditions</u> Let x_* be a local solution such that the linearity independence is satisfied at x_*. Let λ^* such that the KKT conditions are satisfied at (x_*, λ^*) and let $F(\lambda^*)$ be the set

$$ w \in F(\lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x_*)^T w = 0, \; i \in \mathcal{E}, \\ \nabla c_i(x_*)^T w = 0, \; i \in \mathcal{A}(x_*) \cap \mathcal{I}, \; \lambda_i^* > 0 \\ \nabla c_i(x_*)^T w \geq 0, \; i \in \mathcal{A}(x_*) \cap \mathcal{I}, \; \lambda_i^* = 0. \end{cases} $$

Then

- $w^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*) w \geq 0$, for all $w \in F(\lambda^*)$.

<u>2nd-order sufficient conditions</u> Suppose that at some feasible point $x_* \in R^n$ there is a Lagrange multiplier λ^* such that the KKT conditions are satisfied. Then if the following condition is satisfied, then x_* is a strict local minimizer:

- $w^T \nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*) w > 0$ for all $w \in F(\lambda^*)$, $w \neq 0$.

Remarks:
- If $\nabla_{xx}^2 \mathcal{L}(x_*, \lambda^*)$ positive definite, then $Z^T \nabla^2 f(x_*) Z$ is also positive definite (converse not true).
- "Negative definite" instead of "positive definite" for local maximizer.
- The Lagrange multiplier λ_i^* is always 0 for inactive constraints.