Math 273
Homework #4, due on Wednesday, November 25, or on Monday, November 30

[**1**] Consider the quadratic program

$$\min_x q(x) = \frac{1}{2}x^T Gx + x^T d, \text{ subject to } Ax = b,$$

where $G \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $A \in \mathbb{R}^{m\times n}$. Assume that A has full row rank m.

(a) Express the first order necessary conditions for x_* to be a solution in the form of a linear matrix equation in the unknown $(x_* \quad \lambda^*)^T$.

(b) Express in (a) x_* by $x + p$, with x some fixed feasible estimate and unknown $p \in Null(A)$. Re-write the matrix equation now in the unknown $(-p \quad \lambda^*)^T$.

(c) Assume in addition that the reduced-Hessian $Z^T GZ$ is positive definite. Show that the coefficient matrix in (b) is non-singular, thus there is a unique vector pair (x_*, λ^*) satisfying the matrix equation in (a).

[**2**] Show that $(0, -1)^T$ is a local minimizer for the problem
Minimize $f(x) = 2x_1^2 + x_2$ subject to
$x_2 \geq x_1^2 - 1$
$x_1 \geq x_2$.

[**3**] The problem of finding the shortest distance from a point x_0 to the hyperplane $\{x : \ Ax = b\}$ where A has full row rank can be formulated as the quadratic program

$$\min \frac{1}{2}(x - x_0)^T(x - x_0), \quad \text{s.t. } Ax = b.$$

(i) Show that the optimal multiplier is $\lambda^* = (AA^T)^{-1}(b - Ax_0)$, and that the solution is $x_* = x_0 + A^T(AA^T)^{-1}(b - Ax_0)$.

(ii) Show that in the special case where A is a row vector, the shortest distance from x_0 to the solution set of $Ax = b$ is $\frac{|b-Ax_0|}{\|A\|}$.

[**4**] Repeat problem [3], Hw #3 using now Newton's method, and compare the two methods. Give details about your implementation (computation of gradient, of Hessian, of inverse, about α, about your stopping criteria, etc), and include your code.

[**5**] Let V be a real vector space and $F : V \to \overline{\mathbb{R}}$ be a convex function, thus for every $u, v \in \mathcal{V}$, we have $F(\lambda u + (1 - \lambda)v) \leq \lambda F(u) + (1 - \lambda)F(v)$, $\forall \lambda \in [0, 1]$, whenever the RHS is defined (the RHS is not defined when $F(u) = -F(v) = +\infty$ or $F(u) = -F(v) = -\infty$).

(a) If F is convex, show that for every $u_1, ..., u_n$ of points of V and for every family $\lambda_1, ..., \lambda_n$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, we have

$$F(\sum_{i=1}^n \lambda_i u_i) \leq \sum_{i=1}^n \lambda_i F(u_i).$$

(b) If $F : V \to \overline{\mathbb{R}}$ is convex, show that the sections $\{u : \ F(u) \leq a\}$ and $\{u : \ F(u) < a\}$ are convex sets. Show that the converse is not true.

[**6**] The *epigraph* of a function $F : V \to \mathbb{R}$ is the set

$$\text{epi}F = \{(u, a) \in V \times \mathbb{R} | \ f(u) \leq a\},$$

where V is a Banach space. Show that the function F is convex if and only if its epigraph is convex.