Math 273
Homework #4, due on Wednesday, November 25, or on Monday, November 30

[1] Consider the quadratic program
\[
\min_{x} q(x) = \frac{1}{2} x^T G x + x^T d,
\]
subject to \(Ax = b\),
where \(G \in \mathbb{R}^{n \times n}\) is a symmetric matrix, \(A \in \mathbb{R}^{m \times n}\). Assume that \(A\) has full row rank \(m\).
(a) Express the first order necessary conditions for \(x^*\) to be a solution in the form of a linear matrix equation in the unknown \((x^*, \lambda^*)^T\).
(b) Express in (a) \(x\) by \(x + p\), with \(x\) some fixed feasible estimate and unknown \(p \in \text{Null}(A)\). Re-write the matrix equation now in the unknown \((-p \ \lambda^*)^T\).
(c) Assume in addition that the reduced-Hessian \(Z^T G Z\) is positive definite. Show that the coefficient matrix in (b) is non-singular, thus there is a unique vector pair \((x^*, \lambda^*)\) satisfying the matrix equation in (a).

[2] Show that \((0, -1)^T\) is a local minimizer for the problem
Minimize \(f(x) = 2x_1^2 + x_2\) subject to
\[
\begin{align*}
x_2 & \geq x_1^2 - 1 \\
x_1 & \geq 2.
\end{align*}
\]

[3] The problem of finding the shortest distance from a point \(x_0\) to the hyperplane \(\{x : Ax = b\}\) where \(A\) has full row rank can be formulated as the quadratic program
\[
\min \frac{1}{2} (x - x_0)^T (x - x_0), \quad \text{s.t. } Ax = b.
\]
(i) Show that the optimal multiplier is \(\lambda^* = (AA^T)^{-1}(b - Ax_0)\), and that the solution is \(x^* = x_0 + A^T (AA^T)^{-1}(b - Ax_0)\).
(ii) Show that in the special case where \(A\) is a row vector, the shortest distance from \(x_0\) to the solution set of \(Ax = b\) is \(\frac{\|b - Ax_0\|}{\|A\|}\).

[4] Repeat problem [3], Hw #3 using now Newton’s method, and compare the two methods. Give details about your implementation (computation of gradient, of Hessian, of inverse, about your stopping criteria, etc), and include your code.

[5] Let \(V\) be a real vector space and \(F : V \rightarrow \overline{\mathbb{R}}\) be a convex function, thus for every \(u, v \in V\), we have \(F(\lambda u + (1 - \lambda)v) \leq \lambda F(u) + (1 - \lambda)F(v), \forall \lambda \in [0, 1]\), whenever the RHS is defined (the RHS is not defined when \(F(u) = -F(v) = +\infty\) or \(F(u) = -F(v) = -\infty\)).
(a) If \(F\) is convex, show that for every \(u_1, ..., u_n\) of points of \(V\) and for every family \(\lambda_1, ..., \lambda_n\), \(\lambda_i \geq 0\), \(\sum_{i=1}^n \lambda_i = 1\), we have
\[
F(\sum_{i=1}^n \lambda_i u_i) \leq \sum_{i=1}^n \lambda_i F(u_i).
\]
(b) If \(F : V \rightarrow \overline{\mathbb{R}}\) is convex, show that the sections \(\{u : F(u) \leq a\}\) and \(\{u : F(u) < a\}\) are convex sets. Show that the converse is not true.

[6] The epigraph of a function \(F : V \rightarrow \mathbb{R}\) is the set
\[
\text{epi}F = \{(u, a) \in V \times \mathbb{R} | f(u) \leq a\},
\]
where \(V\) is a Banach space. Show that the function \(F\) is convex if and only if its epigraph is convex.