Math 273: Homework #3, due on Monday, November 16

[1] Let Ω be an open and bounded domain in $\mathbb{R}^2$, with sufficiently smooth boundary $\partial\Omega$. Consider the minimization problem in two dimensions

$$\inf_u F(u) = \int_\Omega |Ku - u_0|^2 dxdy + \alpha \int_\Omega f(\nabla u) dxdy,$$

with $u_0 \in L^2(\Omega)$ (square integrable function) a given function, and f is a smooth function on $\mathbb{R}^2$ with real values. Here $K : L^2(\Omega) \to L^2(\Omega)$ is a linear and continuous operator, and its adjoint is K^* (thus K^* has the property $\int_\Omega (Ku)v dxdy = \int_\Omega u(K^* v) dxdy$). Obtain, as in Hw#1[6] and Hw#2[1], the Euler-Lagrange equation associated with the minimization problem that is formally satisfied by a sufficiently smooth optimal u. No explicit boundary conditions are imposed, thus you have to deduce implicit (or natural) boundary conditions on $\partial\Omega$.

[2] (This problem is related with [6] from Hw #1)

Let $u(x, y, t)$ be a smooth solution of the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P),$$

with $u(x, y, 0) = u_0(x, y)$ in Ω and $u(x, y, t) = g(x, y)$ for $(x, y) \in \partial\Omega$ and $t \geq 0$ (recall that P is a notation for $(x, y, u(x, y), u_x(x, y), u_y(x, y))$).

Show that the function $E(t) = F(u(\cdot, \cdot, t))$ is decreasing in time, where $F(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy$.

[3] Apply the gradient descent method described in class to the two-dimensional diffusion problem

$$F(u) = \sum_{1 \leq i,j \leq n} \Big[(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \lambda (u_{i,j} - f_{i,j})^2 \Big],$$

where $f_{i,j}$ is given for $0 \leq i, j \leq n+1$, and with the boundary conditions $u_{i,j} = f_{i,j}$ if $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$. Here $\lambda > 0$ is a tunning parameter. Choose a function f of your choice (for example an image). If you do not have one, you can create a synthetic image. Test various values of the parameter λ and observe the properties of your implementation. Give your choice of the stopping criteria and also plot the value of the objective function versus steps. Plot the data f, your starting point and your final result, as 2D images.

[4] Consider the constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{subject to } Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $m \leq n$, $b \in \mathbb{R}^m$. Tranform the problem into an unconstrained minimization problem.

[5] Verify that the KKT conditions for the bound-constrained problem

$$\min_{x \in \mathbb{R}^n} \phi(x), \text{ subject to } l \leq x \leq u,$$

are equivalent to the compactly stated condition $P_{[l,u]} \nabla \phi(x) = 0$, where the projection operator $P_{[l,u]}$ of a vector $g \in \mathbb{R}^n$ onto the rectangular box $[l, u]$ is defined by

$$(P_{[l,u]} g)_i = \begin{cases} min(0, g_i), & \text{if } x_i = l_i, \\ g_i, & \text{if } x_i \in (l_i, u_i), \text{ for all } i = 1, 2, ..., n \\ max(0, g_i), & \text{if } x_i = u_i. \end{cases}$$