

**Math 273: Homework #2, due on Monday, November 2nd**

[1] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x)) dx,$$

with  $u(x_0) = u_0$ ,  $u(x_1) = u_1$ ,  $u'(x_0) = U_0$ ,  $u'(x_1) = U_1$  given, and  $L$  is a sufficiently smooth function. Obtain the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal  $u$ . Choose test functions  $v$  in  $C^\infty[x_0, x_1]$  that satisfy  $v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0$ , and proceed as in HW1, problem [6] (you should obtain a fourth-order differential equation).

[2] Consider the 1D length functional minimization problem

$$\text{Min}_u F(u) = \int_0^1 L(u'(x)) dx, \text{ or } \text{Min}_u \int_0^1 \sqrt{1 + (u'(x))^2} dx,$$

over functions  $u : [0, 1] \rightarrow \mathbb{R}$  with boundary conditions  $u(0) = 0$ ,  $u(1) = 1$ .

- (a) Find the exact solution of the problem.
- (b) Show that the functional  $u \mapsto F(u)$  is convex.
- (c) Consider a discrete version of the problem: let

$$x_0 = 0 < x_1 < \dots < x_n < x_{n+1} = 1$$

be equidistant points, with  $x_{i+1} - x_i = h$ . For  $\vec{u} = (u_1, \dots, u_n)$ , consider  $f(\vec{u}) = \sum_{i=0}^n \sqrt{1 + \left(\frac{u_{i+1} - u_i}{h}\right)^2}$ , with the additional condition that  $u_0 = 0$  and  $u_{n+1} = 1$ .

Choose an appropriate discretization integer  $n$ . Then numerically and experimentally analyze the behavior of the gradient descent method with backtracking line search. Choose the initial starting point  $u^0$  as a curve joining the points  $(0, 0)$  and  $(1, 1)$ . Record the number of iterations and plot the error  $u^k - u^*$ , where  $u^*$  is the exact minimizer. You could also plot the curve given by  $\vec{u}^k$  at some iterations.

- (d) Repeat question (c), using now Newton's method.
- (e) Discuss the results obtained in (c) and (d).

[3] Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a (linear) self-adjoint operator,  $b \in \mathbb{R}^n$ , and consider the quadratic function for  $x \in \mathbb{R}^n$

$$x \mapsto q(x) := \langle Ax, x \rangle - 2\langle b, x \rangle.$$

Show that the three statements

- (i)  $\inf\{q(x) : x \in \mathbb{R}^n\} > -\infty$
- (ii)  $A \geq O$  and  $b \in \text{Im}A$ .
- (iii) the problem  $\inf\{q(x) : x \in \mathbb{R}^n\} > -\infty$  has a solution

are equivalent. When they hold, characterize the set of minimum points of  $q$ , in terms of the pseudo-inverse of  $A$ .

**Notes:**

• If  $A$  is a symmetric (or self-adjoint) linear operator on  $X$ , then  $\text{Im}A^\perp = \text{Ker}A$ . Let  $p_{\text{Im}A}$  be the operator of orthogonal projection onto  $\text{Im}A$ . For given  $y \in X$ , there is a unique  $x = x(y)$  in  $\text{Im}A$  such that  $Ax = p_{\text{Im}A}y$ . Furthermore, the mapping  $y \mapsto x(y)$  is linear. This mapping is called the pseudo-inverse, or generalized inverse of  $A$ .

• Let  $\Omega$  be an open and bounded subset of  $R^d$ , with Lipschitz-continuous (or sufficiently smooth) boundary  $\partial\Omega$ . Let  $\vec{n} = (n_1, n_2, \dots, n_d)$  be the exterior unit normal to  $\partial\Omega$ . Recall the following fundamental Green's formula, or integration by parts formula: given two functions  $u, v$  (with  $u, v$ , and all their 1st order partial derivatives belonging to  $L^2(\Omega)$ , or  $u, v \in H^1(\Omega)$ ), then

$$\int_{\Omega} uv_{x_i} dx = - \int_{\Omega} u_{x_i} v dx + \int_{\partial\Omega} uv n_i dS.$$