Math 273: Homework #4, due on Monday, November 27

[1] Verify that the KKT conditions for the bound-constrained problem

$$\min_{x\in\mathbb{R}^n} \phi(x), \text{ subject to } l \leq x \leq u,$$

are equivalent to the compactly stated condition $P_{[l,u]}\nabla\phi(x) = 0$, where the projection operator $P_{[l,u]}$ of a vector $g \in \mathbb{R}^n$ onto the rectangular box $[l,u]$ is defined by

$$(P_{[l,u]}g)_i = \begin{cases} min(0, g_i), & \text{if } x_i = l_i, \\ g_i, & \text{if } x_i \in (l_i, u_i), \\ max(0, g_i), & \text{if } x_i = u_i. \end{cases} \text{ for all } i = 1, 2, ..., n$$

(see the 1st order optimality conditions)

[2] Consider the quadratic program

$$\min_x q(x) = \frac{1}{2}x^T Gx + x^T d, \text{ subject to } Ax = b,$$

where $G \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $A \in \mathbb{R}^{m\times n}$. Assume that A has full row rank m.

(a) Express the first order necessary conditions for x_* to be a solution in the form of a linear matrix equation in the unknown $(x_* \;\; \lambda^*)^T$.

(b) Express in (a) x_* by $x + p$, with x some fixed feasible estimate and unknown $p \in Null(A)$. Re-write the matrix equation now in the unknown $(-p \;\; \lambda^*)^T$.

(c) Assume in addition that the reduced-Hessian $Z^T GZ$ is positive definite. Show that the coefficient matrix in (b) is non-singular, thus there is a unique vector pair (x_*, λ^*) satisfying the matrix equation in (a).

[3] Apply the gradient descent method described in class to the two-dimensional diffusion problem

$$F(u) = \sum_{1\leq i,j\leq n} \Big[(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + \lambda(u_{i,j} - f_{i,j})^2\Big],$$

where $f_{i,j}$ is given for $0 \leq i,j \leq n+1$, and with the boundary conditions $u_{i,j} = f_{i,j}$ if $i = 0$ or $i = n+1$ or $j = 0$ or $j = n+1$. Here $\lambda > 0$ is a tunning parameter. Choose a function f of your choice (for example an image). If you do not have one, you can create a synthetic image. Test various values of the parameter λ and observe the properties of your implementation. Give your choice of the stopping criteria and also plot the value of the objective function versus steps. Plot the data f, your starting point and your final result, as 2D images.

[4] The *epigraph* of a function $F : V \to \mathbb{R}$ is the set

$$\text{epi}F = \{(u, a) \in V \times \mathbb{R} |\; f(u) \leq a\},$$

where V is a Banach space. Show that the function F is convex if and only if its epigraph is convex (this is a generalization of the same result in $\mathbb{R}^n$).