Math 273: Homework #3, due on Monday, November 13

NOTES: Please note that exceptionally, there is no class on Wednesday, November 8.

[**1**] Let Ω be an open and bounded domain in R^2, with sufficiently smooth boundary, and consider the minimization problem now in two dimensions

$$\inf_u F(u) = \int_\Omega |k * u - u_0|^2 dx + \alpha \int_\Omega f(\nabla u, u_{xx}, u_{yy}, u_{xy}) dx,$$

with u, u_x and u_y given on the boundary $\partial\Omega$, and $u_0(x, y)$ is a given function. We assume that f is a smooth function and k is a symmetric kernel.

Obtain the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal u. Choose appropriate test functions v in $C^\infty(\Omega)$.

[**2**] Repeat problem [2](c) from Hw #2 with Newton's method.

[**3**] (This problem is related with [1] from Hw #1)

Let $u(x, y, t)$ be a smooth solution of the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P),$$

with $u(x, y, 0) = u_0(x, y)$ in Ω and $u(x, y, t) = g(x, y)$ for $(x, y) \in \partial\Omega$ and $t \geq 0$ (recall that P is a notation for $(x, y, u(x, y), u_x(x, y), u_y(x, y))$).

Show that the function $E(t) = F(u(\cdot, \cdot, t))$ is decreasing in time, where $F(u) = \int_\Omega L(x, y, u, u_x, u_y) dxdy$.

Similarly consider the same PDE satisfied by u, but now with boundary conditions $(L_{u_x}(P), L_{u_y}(P)) \cdot (n_x, n_y) = 0$ on $\partial\Omega$.

[**4**] Show that $\|Bx\| \geq \frac{\|x\|}{\|B^{-1}\|}$ for any nonsingular matrix B.

[**5**] Find the minima of the function $f(x) = x_1 x_2$ on the unit circle $x_1^2 + x_2^2 = 1$. Illustrate this problem geometrically.