Math 273: Homework #2, due on Wednesday, October 25

[1] Consider the minimization problem

$$\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x), u''(x))dx,$$

with \(u(x_0) = u_0, u(x_1) = u_1, u'(x_0) = U_0, u'(x_1) = U_1\) given, and \(L\) is a sufficiently smooth function. Obtain the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal \(u\). Choose test functions \(v\) in \(C^\infty[x_0, x_1]\) that satisfy \(v(x_0) = v(x_1) = v'(x_0) = v'(x_1) = 0\), and proceed as before (you should obtain a fourth-order differential equation).

[2] Consider the 1D length functional

$$\text{Min}_u F(u) = \int_0^1 L(u'(x))dx, \text{ or Min}_u \int_0^1 \sqrt{1 + (u'(x))^2}dx,$$

with boundary conditions \(u(0) = 0, u(1) = 1\).

(a) Find the exact solution of the problem.
(b) Show that the functional \(u \mapsto F(u)\) is convex.
(c) Consider a discrete version of the problem: let \(x_0 = 0 < x_1 < ... < x_n < x_{n+1} = 1\) be equidistant points, with \(x_{i+1} - x_i = h\). For \(\vec{u} = (u_1, ..., u_n)\), consider \(f(\vec{u}) = \sum_{i=0}^{n} \sqrt{1 + \left(\frac{u_{i+1} - u_i}{h}\right)^2}\), with the additional condition that \(u_0 = 0\) and \(u_{n+1} = 1\).

Choose an appropriate discretization integer \(n\) and numerically analyze the behavior of the gradient descent method with backtracking line search. Choose the initial starting point \(u^0\) as a curve joining the points \((0, 0)\) and \((1, 1)\). Record the number of iterations and plot the error \(u_k - u^*\), where \(u^*\) is the exact minimizer. You could also plot the curve given by \(\vec{u}_k\) at some iterations.

**Notes:** Let \(\Omega\) be an open and bounded subset of \(\mathbb{R}^d\), with Lipschitz-continuous (or sufficiently smooth) boundary \(\partial \Omega\). Let \(\vec{n} = (n_1, n_2, ..., n_d)\) be the exterior unit normal to \(\partial \Omega\).

Recall the following fundamental Green’s formula, or integration by parts formula: given two functions \(u, v\) (with \(u, v,\) and all their 1st order partial derivatives belonging to \(L^2(\Omega)\)), then

$$\int_{\Omega} uv_x dx = -\int_{\Omega} u_x v dx + \int_{\partial \Omega} uv_n dS.$$

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