Math 273: Homework #1, due on Monday, October 9

1. Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Show that $x^* = (1, 1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

2. Let a be a given n-vector, and A be a given $n \times n$ symmetric matrix. Compute the gradient and Hessian of $f_1(x) = a^T x$ and $f_2(x) = x^T A x$.

3. Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

4. Suppose that $\hat{f}(z) = f(x)$, where $x = Sz + s$ for some $S \in R^{n \times n}$ and $s \in R^n$. Show that

$$\nabla \hat{f}(z) = S^T \nabla f(x), \qquad \nabla^2 \hat{f}(z) = S^T \nabla^2 f(x) S.$$

5. (Directional derivative) Let $f : R^n \to R$ be continuously differentiable. Show that

$$\lim_{\epsilon \to 0} \frac{f(x + \epsilon p) - f(x)}{\epsilon} = \nabla f(x)^T p.$$

6. Consider the minimization problem

$$\inf_u \int_\Omega L(x, y, u, u_x, u_y) dx dy, \qquad u = g \text{ on } \partial\Omega,$$

where g is given. Let Ω be a bounded region in R^2. Show that a sufficiently smooth solution u satisfies the Euler equation

$$\frac{\partial}{\partial x} L_{u_x}(P) + \frac{\partial}{\partial y} L_{u_y}(P) - L_u(P) = 0$$

on Ω, where $P = (x, y, u(x, y), u_x(x, y), u_y(x, y))$.

Apply the above result to the case when $L(x, y, u_x, u_y) = u_x^2 + u_y^2 - 2fu$.