Connections with the finite dimensional case and with the KKT conditions

Example: Nonlinear inequality constrained optimization

Let $V = V^* = \mathbb{R}^n$, $Y = Y^* = \mathbb{R}^m$. Assume $f : \mathbb{R}^n \to \mathbb{R}$ convex and l.s.c. Let functions $c_1, ..., c_m : \mathbb{R}^n \to \mathbb{R}$ be convex and l.s.c. Assume that there is a $u_0 \in \mathbb{R}^n$ such that $c_i(u_0) \leq 0$ for all $i = 1, ..., m$. Denote by $c(u) = (c_1(u), ..., c_m(u))$.

The primal problem is

$$\inf_{u \in \mathbb{R}^n,\ c_i(u) \leq 0, i=1,\ldots,m} f(u).$$

The perturbation is defined by

$$\Phi(u,p) = \begin{cases} f(u) \text{ if } c(u) \leq p \\ +\infty \text{ otherwise} \end{cases},$$

or

$$\Phi(u,p) = \begin{cases} f(u) \text{ if } c_i(u) \leq p_i,\ i = 1, ..., m \\ +\infty \text{ otherwise} \end{cases}.$$

Then $\Phi(u,p)$ can also be expressed as $\Phi(u,p) = f(u) + \chi_{E_p}(u)$, where χ_{E_p} is the indicator function of the set

$$E_p = \{u \in \mathbb{R}^n :\ c(u) \leq p\}.$$

It is possible to show that the dual problem is

$$\sup_{p^* \leq 0} \inf_{u \in \mathbb{R}^n} \Big\{ - < p^*, c(u) > + f(u) \Big\}.$$

Also, it is possible to show that the Lagrangian defined by

$$-L(u,p^*) := \sup_{p \in \mathbb{R}^m} \{ < p^*, p > - \Phi(u,p) \}$$

becomes

$$L(u,p^*) = f(u) - \sum_{i=1}^m p_i^* c_i(u) \text{ if } p_i^* \leq 0.$$

Then we have the KKT theorem:

Theorem: $\bar{u}$ is a solution of the primal problem if and only if there is a $\bar{p}^* \in \mathbb{R}^m$, $\bar{p}^* \leq 0$ such that

$$L(\bar{u},p) \leq L(\bar{u},\bar{p}^*) \leq L(u,\bar{p}^*) \text{ for all } u \in \mathbb{R}^n,\ p \in R^m, p \leq 0,$$

and the extremality relation holds $< \bar{p}^*, c(\bar{u}) >= 0$, which implies that for all i, $1 \leq i \leq m$,

$$\text{either } \Big\{ c_i(\bar{u}) < 0 \text{ and } p_i^* = 0 \Big\} \text{ or } \Big\{ c_i(\bar{u}) = 0 \text{ and } \bar{p}_i^* < 0 \Big\}.$$

Remark: The usual notations in the finite dimensional case where $f(u) = f(x)$, for $u = x \in \mathbb{R}^n$, $p = \lambda \in \mathbb{R}^m$, and we substituted $C_i(x) \geq 0$ by $-C_i(x) = c_i(x) \leq 0$. Thus the Lagrange multiplier $\bar{p}^* = \lambda^*$ here is negative, while with $C_i(x) \geq 0$ we have $\bar{p}^* = \lambda^* \geq 0$.