Math 270C: Assignment #7
(last assignment)

Due on Friday, June 13 (under the door of my office, with Babette Dalton in MS 7619, or in my mailbox).

[1] Show the main intermediate steps of the Buneman's Algorithm when applied to a non-singular 7×7 block-system $Mz = b$, with $q_0 = q = 2^{2+1} - 1$, and with M having the matrix A on its main block-diagonal and the identity matrix I on the two off-block-diagonals:

$$M = \begin{pmatrix} A & I & O & O & O & O & O \\ I & A & I & O & O & O & O \\ O & I & A & I & O & O & O \\ O & O & I & A & I & O & O \\ O & O & O & I & A & I & O \\ O & O & O & O & I & A & I \\ O & O & O & O & O & I & A \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}.$$

[2] Give the analytic solution to

$$-(3u_{xx} + 2u_{yy}) = sin(\frac{3}{2}\pi x)sin(4\pi y) + sin(\frac{5}{2}\pi x)sin(2\pi y),$$

for $(x, y) \in [0, 2] \times [0, 1]$ and $u(x, y) = 0$ on the boundary of this domain.

[3] Suppose a uniform mesh that has M panels in the x direction and N panels in the y direction for the domain $[0, 2] \times [0, 1]$ is used. If u_{xx} and u_{yy} are approximated using standard 2nd order, 3-point difference formulas, what is the solution of the discrete equations that approximates the problem in [2] ?

[4] Use preconditioned conjugate gradients to compute the solution to the linear set of equations arising from a discretization of

$$\text{div}(a(x, y)\nabla u) = f \text{ for } (x, y) \in \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

where $\Omega = [-1, 1] \times [-1, 1]$, $f(x, y) = sin(\pi x) \sin(\pi y)$, and $a(x, y)$ is a discontinuous function given by

$$a(x, y) = \begin{cases} \alpha_0 \text{ if } \sqrt{x^2 + y^2} \leq \frac{1}{2}, \\ \alpha_1 \text{ if } \sqrt{x^2 + y^2} > \frac{1}{2}. \end{cases}$$

[a] Give a 5-point, second order, symmetric discretization of the variable coefficient operator $\mathrm{div}(a(x,y)\nabla u)$.

[b] Use values of $\alpha_0 = 1$ and $\alpha_1 = 10,\ 100,\ 1000$. Use 100 panels in each direction and stop your iteration when the relative size of the residual is $\leq \frac{h^2}{\|b\|}$. Record the number of iterations required for each set of coefficients.

[c] Use values of $\alpha_0 = 10,\ 100,\ 1000$ and $\alpha_1 = 1$. Use 100 panels in each direction and stop your iteration when the relative size of the residual is $\leq \frac{h^2}{\|b\|}$. Record the number of iterations required for each set of coefficients.

[d] Use values of $\alpha_0 = 1$ and $\alpha_1 = 1000$. Use 200 panels in each direction and stop your iteration when the relative size of the residual is $\leq \frac{h^2}{\|b\|}$. Record the number of iterations required.

[e] How does the number of iterations depend upon the ratio $\frac{\alpha_0}{\alpha_1}$?

[f] How does the number of iterations depend upon the mesh size ?

If you wish, you can plot the size of the relative residual versus iterations, and the final solution as a surface using Matlab together with its level lines.

You can work with the true residual size $\|b - Ax^k\|$ instead of the relative residual size (then you may want to stop if the residual size is $\leq h^2$); or you can work with the "relative" residual size, that can be taken $\frac{\|b-Ax^k\|}{\|b-Ax^0\|}$. You may choose $x^0 = 0$.