Math 270C: Assignment #7
(last assignment)

Due on Friday, June 13 (under the door of my office, with Babette Dalton in MS 7619, or in my mailbox).

[1] Show the main intermediate steps of the Buneman’s Algorithm when applied to a non-singular 7 × 7 block-system \( Mz = b \), with \( q_0 = q = 2^{2^1 - 1} \), and with \( M \) having the matrix \( A \) on its main block-diagonal and the identity matrix \( I \) on the two off-block-diagonals:

\[
M = \begin{pmatrix}
A & I & O & O & O & O & O \\
I & A & I & O & O & O & O \\
O & I & A & I & O & O & O \\
O & O & I & A & I & O & O \\
O & O & O & I & A & I & O \\
O & O & O & O & I & A & I \\
O & O & O & O & O & I & A
\end{pmatrix}
\cdot
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
z_6 \\
z_7
\end{pmatrix}
= \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6 \\
b_7
\end{pmatrix}.
\]

[2] Give the analytic solution to

\[-(3u_{xx} + 2u_{yy}) = \sin\left(\frac{3}{2}\pi x\right)\sin(4\pi y) + \sin\left(\frac{5}{2}\pi x\right)\sin(2\pi y),\]

for \((x, y) \in [0, 2] \times [0, 1]\) and \(u(x, y) = 0\) on the boundary of this domain.

[3] Suppose a uniform mesh that has \( M \) panels in the \( x \) direction and \( N \) panels in the \( y \) direction for the domain \([0, 2] \times [0, 1]\) is used. If \( u_{xx} \) and \( u_{yy} \) are approximated using standard 2nd order, 3-point difference formulas, what is the solution of the discrete equations that approximates the problem in [2]?

[4] Use preconditioned conjugate gradients to compute the solution to the linear set of equations arising from a discretization of

\[
\text{div}(a(x, y)\nabla u) = f \quad \text{for} \quad (x, y) \in \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial\Omega,
\]

where \( \Omega = [-1, 1] \times [-1, 1] \), \( f(x, y) = \sin(\pi x)\sin(\pi y) \), and \( a(x, y) \) is a discontinuous function given by

\[
a(x, y) = \begin{cases}
\alpha_0 & \text{if } \sqrt{x^2 + y^2} \leq \frac{1}{2}, \\
\alpha_1 & \text{if } \sqrt{x^2 + y^2} > \frac{1}{2}.
\end{cases}
\]
[a] Give a 5-point, second order, symmetric discretization of the variable coefficient operator \( \text{div}(a(x, y) \nabla u) \).

[b] Use values of \( \alpha_0 = 1 \) and \( \alpha_1 = 10, \ 100, \ 1000 \). Use 100 panels in each direction and stop your iteration when the relative size of the residual is \( \leq \frac{h^2}{\|b\|} \). Record the number of iterations required for each set of coefficients.

[c] Use values of \( \alpha_0 = 10, \ 100, \ 1000 \) and \( \alpha_1 = 1 \). Use 100 panels in each direction and stop your iteration when the relative size of the residual is \( \leq \frac{h^2}{\|b\|} \). Record the number of iterations required for each set of coefficients.

[d] Use values of \( \alpha_0 = 1 \) and \( \alpha_1 = 1000 \). Use 200 panels in each direction and stop your iteration when the relative size of the residual is \( \leq \frac{h^2}{\|b\|} \). Record the number of iterations required.

[e] How does the number of iterations depend upon the ratio \( \frac{\alpha_0}{\alpha_1} \)?

[f] How does the number of iterations depend upon the mesh size?

If you wish, you can plot the size of the relative residual versus iterations, and the final solution as a surface using Matlab together with its level lines.

You can work with the true residual size \( \|b - Ax^k\| \) instead of the relative residual size (then you may want to stop if the residual size is \( \leq h^2 \)); or you can work with the “relative” residual size, that can be taken \( \frac{\|b - Ax^k\|}{\|b - Ax^0\|} \). You may choose \( x^0 = 0 \).