Math 270C: Assignment #3

Due on Wednesday, April 30.

[**1**] Show that if $A = M - N$ is singular, then we can never have $\rho(M^{-1}N) < 1$ even if M is non-singular.

[**2**] Show that the Gauss-Jacobi iteration converges for 2-by-2 symmetric positive definite systems.

[**3**] Consider the 2-by-2 matrix

$$A = \begin{bmatrix} 1 & \rho \\ -\rho & 1 \end{bmatrix}.$$

(a) Under what conditions will Gauss-Seidel converge with this matrix ?
(b) For what range of ω will the SOR method converge ?

[**4**] Consider the Poisson equation

$$-(u_{xx} + u_{yy}) = f(x, y), \quad \text{for } (x, y) \in \Omega = (0, 1) \times (0, 1),$$

$$u(x, y) = \phi(x, y), \text{ on } \partial\Omega.$$

The grid points are $(x_i, y_j) = (ih, jh)$, for $0 \leq i, j \leq N$, where $h = 1/N$. Assume $f_{i,j} = -4$, $\phi(x, y) = x^2 + y^2$. The exact solution is $u(x, y) = x^2 + y^2$, or $u_{i,j} = (i^2 + j^2)h^2$. As a starting value, $u^0_{i,j} = 0$ is used. An iterative method will compute an approximation $u^{(k)}_{i,j}$, $k = 0, 1, 2, ...$Using the lexicographical ordering, implement the Gauss-Seidel and the SOR methods, with $h = 1/32$. For the SOR method, use an "optimal" value $\omega = 1.821465$.

Define the error

$$E^{(k)} := max\{|u^{(k)}_{i,j} - (i^2 + j^2)h^2| : \quad 1 \leq i, j \leq N - 1\}.$$

Plot the evolution function of iterations of the errors for both methods. For G-S use 300 iterations, while for SOR use 150 iterations.