Math 270C: Assignment #1

Due on Wednesday, April 9.

[**1**] Let A be the $(N-1)\times(N-1)$ matrix associated with a finite difference solution to the two-point BVP:

$$-u_{xx} = f(x),\ u(0) = u(1) = 0$$

constructed using equispaced grid points in $[0,1]$ and the standard three-point, second order, discretization to approximate u_{xx} (let $x_0 = 0$ and $x_N = 1$).

(a) Verify (analytically) that the $N-1$ eigenvectors of A are given by the discrete sin functions: i.e. vectors whose elements consist of $\sin(k\pi x_j)$ for a value of k in $[1, N-1]$ and $x_j = \frac{j}{N}$. Specifically, the jth element of s^k is given by $s_j^k = \sin(k\pi x_j)$.

(b) Give a formula for the eigenvalues of A.

(c) Give the analytical solution to $Ax = f$ when f is the vector obtained by evaluating $-\sin(\pi x)$ at the nodes $x_j = \frac{j}{N}$ for $j = 1, ..., N-1$.

Hint: For (b) and (c), you can use the complex exponential relation

$$\sin(k\pi x_j) = \frac{e^{ik\pi x_j} - e^{-k\pi x_j}}{2i}$$

for easier manipulations.

[**2**] Create a program that uses Gauss-Jacobi to iteratively compute a solution to $Au = f$ with A and f given as above. Use an initial guess $u_0 = \vec{0}$, $h = \frac{1}{20}$. How many iterations are needed to obtain the 2-norm of the residual of size h^2 ? Do the result change if you use an initial guess $u_0 = f$?

[**3**] Create a program that uses Gauss-Seidel to iteratively compute a solution to $Au = f$ with A and f given as above. Use an initial guess $u_0 = \vec{0}$, $h = \frac{1}{20}$. How many iterations are needed to obtain the 2-norm of the residual of size h^2 ? Do the result change if you use an initial guess $u_0 = f$?

What you should turn in: the solutions to the theoretical questions, the listings of your programs for computational problems and the number of iterations you recorded (recall the name of the method and the value of h). Any other optional plots will be helpful.