More definitions and properties of matrices

$A \in \mathcal{M}(n \times n)$ is a real or complex matrix

Definitions and notations

A^T = transpose of A

$A^H = \bar{A}^T$ = complex conjugate and transpose, for complex matrices

If $A = A^T$ we say that A is symmetric. If $A = A^H$ we say that A is Hermitian.

Unitary or orthogonal matrix Q if $Q^T Q = I$ for real matrices ($Q^T = Q^{-1}$) or if $Q^H Q = I$ for complex matrices ($Q^H = Q^{-1}$).

Properties:

If A real, then $\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(AA^T)}$.

If A complex, then $\|A\|_2 = \sqrt{\rho(A^H A)} = \sqrt{\rho(AA^H)}$.

Assume B is unitary or orthogonal. Then eigenvalues of A coincide with eigenvalues of $B^{-1}AB$ (or $\sigma(A) = \sigma(B^{-1}AB)$).

$(AB)^T = B^T A^T$

$(AB)^H = B^H A^H$.

If $T = Q^H A Q$ with $Q^H = Q^{-1}$, then $\|A\|_2 = \|T\|_2$.

For matrix norms: $\|AB\| \leq \|A\| \cdot \|B\|$.

Let $< \cdot, \cdot >$ be the Euclidean scalar product.

Definitions:

We say that A is positive definite ($A > O$) if $< Ax, x >> 0$ for any vector $x \neq \vec{0}$.

We say that A is positive semi-definite ($A \geq O$) if $< Ax, x >\geq 0$ for any vector x.

We write $A > B$ iff $A - B > O$ and $A \geq B$ iff $A - B \geq O$.

Properties:

A is positive definite iff all its eigenvalues are strictly positive.

$A > B \;\Leftrightarrow\; CAC^H > CBC^H$, for all C regular ($det(C) \neq 0$) (complex case)

$A > B \;\Leftrightarrow\; CAC^T > CBC^T$, for all C regular ($det(C) \neq 0$) (real case).

$A > O \Rightarrow D = diag(A) > O$.

$A > O$, $B > O \Rightarrow A + B > O$.

$\xi I < A < \eta I \Rightarrow \sigma(A) \subset (\xi, \eta)$ (if A Hermitian or symmetric).

Assume $D = diag(d_{ii})$ = dialgonal matrix, with $d_{ii} > 0$. Then $D > O$, and $D^{1/2} = Diag(\sqrt{d_{ii}})$, $D^{-1/2} = \left(D^{1/2}\right)^{-1}$.

Let $A > O$ be positive definite. Then its diagonal matrix $D = Diag(a_{ii}) > O$.

If A positive definite (or positive semi-definite), then $A^{1/2}$ is positive definite (or positive semi-definite). $A^{1/2}$ is well defined by the unique (positive definite or positive semi-definite) solution of $X^2 = A$.

$A^{-1/2}$ represents $\left(A^{1/2}\right)^{-1}$.