Math 270C: Assignment 5
Due Wednesday, February 22, 2006 (late homework accepted)
Note: Monday, February 20, there is no class due to national holiday.

[**1**] Assume that the $n \times n$, symmetric, positive definite matrix A has the eigenvalues λ_i, $i = 1, ..., n$, with corresponding linearly independent eigenvectors v_i, $i = 1, ..., n$.

(a) Find the eigenvalues of $P_k(A)$, where $P_k(A) = \gamma_0 I + \gamma_1 A + ... + \gamma_k A^k$, with γ_i constants.

(b) Show that any eigenvector v_i of A is also an eigenvector of $P_k(A)$.

(c) Assume that $x \in R^n$ is given by $x = \sum_{i=1}^{n} \xi_i v_i$, for some constants ξ_i. Assume now that the eigenvectos v_i are orthonormal. Show that

$$\|x\|_A^2 = \sum_{i=1}^{n} \lambda_i \xi_i^2,$$

where $\|x\|_A = \sqrt{< Ax, x >}$.

[**2**] Let $\{\lambda_i, v_i\}$, $i = 1, ..., n$ be the eigenpairs of A. Show that the eigenvalues and eigenvectors of

$$[I + P_k(A)A]^T A [I + P_k(A)A]$$

are $\lambda_i [1 + \lambda_i P_k(\lambda_i)]^2$ and v_i, respectively, where P_k is defined as before.

[**3**] Assume true the following theorem: *Suppose $A \in R^{n \times n}$ is symmetric, positive definite, and $b \in R^n$. If x^k is the iteration given by the gradient descent method, and if $K = K_2(A) = \frac{\lambda_{max}}{\lambda_{min}}$ is the 2-condition number of A, then $\|x - x^k\|_A \leq 2\|x - x^0\|_A \left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^k$.*

Using this theorem, show that

$$\|x^k - A^{-1}b\|_2 \leq 2\sqrt{K}\left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^k \|x^0 - A^{-1}b\|_2.$$

[**4**] Implement the conjugate gradient method for the model problem from [4], assignment 3. Use again $h = 1/32$. Plot the error versus iterations. Comment your results.