Assume that the $n \times n$, symmetric, positive definite matrix $A$ has the eigenvalues $\lambda_i$, $i = 1, ..., n$, with corresponding linearly independent eigenvectors $v_i$, $i = 1, ..., n$.

(a) Find the eigenvalues of $P_k(A)$, where $P_k(A) = \gamma_0 I + \gamma_1 A + ... + \gamma_k A^k$, with $\gamma_i$ constants.

(b) Show that any eigenvector $v_i$ of $A$ is also an eigenvector of $P_k(A)$.

(c) Assume that $x \in \mathbb{R}^n$ is given by $x = \sum_{i=1}^{n} \xi_i v_i$, for some constants $\xi_i$. Assume now that the eigenvectors $v_i$ are orthonormal. Show that

$$\|x\|_A^2 = \sum_{i=1}^{n} \lambda_i \xi_i^2,$$

where $\|x\|_A = \sqrt{\langle Ax, x \rangle}$.

Let $\{\lambda_i, v_i\}, i = 1, ..., n$ be the eigenpairs of $A$. Show that the eigenvalues and eigenvectors of

$$[I + P_k(A)A]^T A[I + P_k(A)A]$$

are $\lambda_i[1 + \lambda_i P_k(\lambda_i)]^2$ and $v_i$, respectively, where $P_k$ is defined as before.

Assume true the following theorem: Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric, positive definite, and $b \in \mathbb{R}^n$. If $x^k$ is the iteration given by the gradient descent method, and if $K = K_2(A) = \frac{\lambda_{\max}}{\lambda_{\min}}$ is the 2-condition number of $A$, then

$$\|x - x^k\|_A \leq 2\|x - x^0\|_A \left(\frac{\sqrt{K} - 1}{\sqrt{K} + 1}\right)^k.$$

Using this theorem, show that

$$\|x^k - A^{-1}b\|_2 \leq 2\sqrt{K} \left(\frac{\sqrt{K} - 1}{\sqrt{K} + 1}\right)^k \|x^0 - A^{-1}b\|_2.$$

Implement the conjugate gradient method for the model problem from [4], assignment 3. Use again $h = 1/32$. Plot the error versus iterations. Comment your results.