NOTE:

• Exceptionally, there is no class on Monday, January 24. There will be the usual class meetings on Wednesday and Friday of the same week.

• Deadline extension for Assignment 2: due on Wednesday, January 26.

Math 270C: Assignment 3

Due Wednesday, February 2, 2005

[1] For an $n \times n$ matrix A, consider the iterative method

$$\vec{x}^{(k+1)} = M^{-1}N\vec{x}^{(k)} + M^{-1}\vec{b},$$

based upon the splitting of A: $A = M - N$. Show that if A is singular, then $\rho(M^{-1}N) \geq 1$.

[2] Let A be an $n \times n$ symmetric positive definite matrix. Consider the $A - norm$ of a vector $\vec{x} \in R^n$ defined by

$$\|\vec{x}\|_A^2 = < \vec{x}, A\vec{x} > .$$

(a) If A is not positive definite, what conditions that define a vector norm are not satisfied by $\|\vec{x}\|_A$?

(b) Is symmetry required to insure that $\|\vec{x}\|_A$ is a vector norm ?

[3] Assume A is an $n \times n$ symmetric positive definite matrix. Let $\{\vec{p}_j\}$, $j = 1...M$ be a set of $A-$orthogonal vectors, e.g. $< \vec{p}_j, A\vec{p}_i >= 0$ if $i \neq j$. Show that if $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$, then the coefficients $\{c_j\}$, $j = 1...M$ that minimize

$$\|\vec{x} - \sum c_j\vec{p}_j\|_A$$

are given by

$$c_j = \frac{< \vec{b}, \vec{p}_j >}{< \vec{p}_j, A\vec{p}_j >}.$$

(Tip: $\vec{x} = A^{-1}\vec{b}$)

[4] Consider the Poisson equation

$$-(u_{xx} + u_{yy}) = f(x, y), \quad \text{for } (x, y) \in \Omega = (0, 1) \times (0, 1),$$

$$u(x, y) = \phi(x, y), \text{ on } \partial\Omega.$$

The grid points are $(x_i, y_j) = (ih, jh)$, for $0 \le i, j \le N$, where $h = 1/N$. Assume $f_{i,j} = -4$, $\phi(x, y) = x^2 + y^2$. The exact solution is $u(x, y) = x^2 + y^2$, or $u_{i,j} = (i^2+j^2)h^2$. As a starting value, $u^0_{i,j} = 0$ is used. An iterative method will compute an approximation $u^{(k)}_{i,j}$, $k = 0, 1, 2, ...$Using the lexicographical ordering, implement the Gauss-Seidel and the SOR methods, with $h = 1/32$. For the SOR method, use an "optimal" value $\omega = 1.821465$.

Define the error

$$E^{(k)} := max\{|u^{(k)}_{i,j} - (i^2 + j^2)h^2| : \quad 1 \le i, j \le N - 1\}.$$

Plot the evolution function of iterations of the errors for both methods. For G-S use 300 iterations, while for SOR use 150 iterations.