Math 270C: Assignment 2
Deadline extension: Due Wednesday, January 26, 2005

[1] Let $D_h$ represent the discrete Laplace operator in two dimensions arising from the discretization of

$$-(u_{xx} + u_{yy}) = f(x, y), \ (x, y) \in \Omega = (0, 1)^2, \ u(x, y) = 0 \ \text{if} \ (x, y) \in \partial \Omega,$$

with the standard five-point, second order, discretization to approximate $u_{xx} + u_{yy}$. Assume that $h = \Delta x = \Delta y$, with $x_0 = 0, x_j = jh, \ j = 0, ..., N, \ x_N = 1$, and $y_0 = 0, y_j = jh, \ j = 0, ..., N, \ y_N = 1$.

(a) Verify (analytically) that the $(N - 1) \times (N - 1)$ eigenvectors of $D_h$ are the discrete functions $s^{(k_1, k_2)}$, whose $(i, j)$th entry is given by $\sin(k_1 \pi x_i) \sin(k_2 \pi y_j)$ for a value of $k_1$ in $\{1, ..., N - 1\}$ and $k_2$ in $\{1, ..., N - 1\}$, and $x_i = i(1/N), \ y_j = j(1/N)$.

(b) Give a formula for the eigenvalues of $D_h$.

[2] Show that the Gauss-Jacobi iteration can be written in the form $x^{(k+1)} = x^{(k)} + Hr^{(k)}$ where $r^{(k)} = b - Ax^{(k)}$. Repeat for the Gauss-Seidel iteration.

[3] Show that if $A$ is strictly diagonally dominant, then the Gauss-Seidel iteration converges.