

Math 269C. L. Vese. HW#5.

[1] Let $I = [0, h]$ and let $\pi v \in P_1(I)$ be the linear interpolant that agrees with $v \in C^2(I)$ at the end points of I . Using the technique of the proof of Thm. 4.1, prove estimates for $\|v - \pi v\|_{L^\infty(I)}$ and $\|v' - (\pi v)'\|_{L^\infty(I)}$, cf. (1.12) and (1.13).

[2] Consider the PDE (in the distributional sense)

$$-\Delta u + k^2 u = f \quad \text{in } R^n,$$

where k is a constant. Let $s \in R$. Show that, for all $f \in H^s(R^n)$, there exists a unique $u \in H^{s+2}(R^n)$, solution of the PDE, with $k \in R$, $k \neq 0$.

Hint: use the Fourier transform (see handout of useful results for notations of Sobolev spaces with arbitrary exponent s).

Optional Problems

[3] The following elliptic problem is approximated by the finite element method,

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u(x)) &= f(x), \quad x \in \Omega \subset R^2, \\ u(x) &= 0, \quad x \in \partial\Omega_1, \\ \frac{\partial u(x)}{\partial x_1} + u(x) &= 0, \quad x \in \partial\Omega_2, \\ \frac{\partial u(x)}{\partial x_2} &= 0, \quad x \in \partial\Omega_3, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < 1\}, \\ \Gamma_1 = \partial\Omega_1 &= \{(x_1, x_2) : x_1 = 0, 0 \leq x_2 \leq 1\}, \\ \Gamma_2 = \partial\Omega_2 &= \{(x_1, x_2) : x_1 = 1, 0 \leq x_2 \leq 1\}, \\ \Gamma_3 = \partial\Omega_3 &= \{(x_1, x_2) : 0 < x_1 < 1, x_2 = 0, 1\}, \end{aligned}$$

and

$$0 < A \leq a(x) \leq B.$$

- (a) Determine an appropriate weak formulation of the problem.
- (b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness and for the convergence of a finite element method (assume $f \in L^2(\Omega)$, $a \in L^\infty(\Omega)$).
- (c) Describe briefly a finite element mesh, a FEM using P_1 elements, and a set of basis functions such that the linear system from the finite element approximation is sparse and of band structure.

[4] (a) Develop and describe the piecewise linear Galerkin finite element approximation of,

$$\begin{aligned}
 -\nabla \cdot (a(x)\nabla u) + b(x)u &= f(x), & x &= (x_1, x_2) \in \Omega, \\
 u &= 2, & x &\in \partial\Omega_1, \\
 \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u &= 2, & x &\in \partial\Omega_2,
 \end{aligned}$$

where $f \in L^2(\Omega)$,

$$\begin{aligned}
 \Omega &= \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}, \\
 \partial\Omega_1 &= \{x \mid x_1 = 0, 0 \leq x_2 \leq 1\} \cup \{x \mid x_2 = 0, 0 \leq x_1 \leq 1\}, \\
 \partial\Omega_2 &= \{x \mid x_1 > 0, x_2 > 0, x_1 + x_2 = 1\}, \\
 0 < a &\leq a(x) \leq A, 0 < b \leq b(x) \leq B.
 \end{aligned}$$

(b) Justify the approximation by analyzing the appropriate bilinear and linear forms. Give a convergence estimate and quote the appropriate theorems for convergence.