Math 269c. Vese. HW #3

[1] This question corresponds to the 2D model problem. Find the linear basis functions for the triangle K with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Using this result, show that the linear system (1.25) of Example 1.1. (textbook) has the stated form (pages 30-31) (there is a typo in the textbook regarding the values of the global stiffness matrix for this case). If you do not have the textbook, please let me know and I can copy the necessary page.

[2] Consider the problem with an inhomogeneous boundary condition,

$$\begin{cases} -\triangle u = f \text{ in } \Omega, \\ u = u_0 \text{ on } \Gamma = \partial\Omega, \end{cases}$$

where f and u_0 are given. Briefly show that this problem can be given the following equivalent formulations:

(V) Find $u \in V(u_0)$ such that $a(u,v) = (f,v)$, $\forall v \in H_0^1(\Omega)$,

(M) Find $u \in V(u_0)$ such that $F(u) \leq F(v)$, $\forall v \in V(u_0)$,

where $V(u_0) = \{v \in H^1(\Omega) : \ v = u_0 \text{ on } \Gamma\}$ and F is defined in the usual way, $F(v) = \frac{1}{2}a(v,v) - (f,v)$.

Then formulate a finite element method and prove an error estimate (as in Thm. 1.1, page 24).

Recall: $H_0^1(\Omega) = \{v \in L^2(\Omega), \ \nabla v \in L^2(\Omega)^n, \ v = 0 \text{ on } \partial\Omega\}$, where n is the spatial dimension.

[3] (a) Give a weak variational formulation of the problem

$$\frac{d^4u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied. Which boundary conditions are essential and which are natural ?

(b) Solve the same problem with the following alternative boundary conditions:

$$u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where γ is a positive constant.

[**4**] Give a weak variational formulation of the Robin's problem

$$-\triangle u = f \quad \text{in } \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$

where γ is a constant. When are the assumptions of the Lax-Milgram Lemma satisfied ?

[**5**] Consider the Neumann problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma = \partial\Omega, \\ \int_\Omega u(x)dx &= 0. \end{aligned}$$

where $f : \Omega \to R$ and $g : \partial\Omega \to R$ satisfy the *compatibility condition*

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(x)d\sigma(x) = 0.$$

(a) Why condition "$\int_\Omega u(x)dx = 0$" was added here ? Why do we need the compatibility condition ?

(b) Give a weak variational formulation of the problem, and prove that the conditions of the Lax-Milgram Lemma are satisfied, under the necessary assumptions on f and g that you would specify.