Math 269C: Vese HW#2

[1] Let V be a Hilbert space and $L : V \to R$ be a linear form. Show that L is bounded if and only if L is continuous. Then directly state the corresponding result for a bilinear form $a : V \times V \to R$.

[2] Let V be a Hilbert space and the (nonlinear) operator $A : V \to V$, satisfying

(i) there is $M \geq 0$ s.t. $\forall\ u, v \in V$, $\|Au - Av\| \leq M\|u - v\|$

(ii) there is $\alpha > 0$ s.t. $\forall\ u, v \in V$, $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$

Show that the nonlinear equation $Au = f$ has a unique solution (for some $f \in V$), using the Banach fixed point theorem and the same technique for proving the Lax-Milgram theorem (introduce the function g_λ).

[3] Let V be a complex Hilbert space, $a : V \times V \to C$ a sesquilinear form, $L : V \to C$ an anti-linear form. Assume that a is Hermitian, thus $a(v, u) = \overline{a(u, v)}$, $\forall u, v \in V$, and that $a(v, v) \geq 0$. Consider the problems

$$\text{(V) Find } u \in V \text{ s.t. } a(u, v) = L(v),\ \forall v \in V,$$

$$\text{(M) Find } u \in V \text{ s.t. } J(u) = \inf_{v \in V} J(v),$$

with $J : V \to R$ defined by $J(v) = \frac{1}{2}a(v, v) - \mathrm{Re}L(v)$.

Show that $u \in V$ is solution of (V) iff $u \in V$ is solution of (M).

• **Note:** Over the complex space C, with the above notations, we say that $a : V \times V \to C$ is a sesquilinear form if a is linear in the first argument (under the usual addition and scalar multiplication) and antilinear in the second argument (usual addition but $a(u, \lambda v) = \bar{\lambda} a(u, v)$). Note that if $\mathrm{Re}a(v, v) \geq \alpha\|v\|^2$, together with a and L bounded, then the Lax-Milgarm Lemma holds in the complex case with the same proof.