Math 269C Spring. Instructor: L. Vese. HW #1.

[**1**] Consider the two-point boundary value problem

$$-u''(x) = x^2 - 3x + 2 \quad \text{for } x \in (0,1),$$

$$u(0) = u(1) = 0.$$

Find the exact (unique) solution of the problem (by integrating twice).

[**2**] Let $V = H_0^1(0,1)$, $f \in L^2(0,1)$, and the 1D model problem:

$$\text{(D)} \qquad -u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

(a) Construct a finite-dimensional subspace V_h of V consisting of continuous functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions ? (see lecture notes or Section 3.2 from the textbook by Johnson).

(b) Find the corresponding basis functions.

(c) Formulate a finite element method for (D) using the space V_h and write down the corresponding linear system of equations in the case of a uniform partition.

[**3**] Consider the BVP

$$\frac{d^4u}{dx^4} = f, \quad 0 < x < 1,$$

$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

for $f \in L^2(0,1)$.

(a) Show that this problem can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v), \quad \forall v \in W,$$

where $W = H_0^2(0,1)$ (functions in $H^2(0,1)$ with $v(0) = v'(0) = v(1) = v'(1) = 0$). Recall that $(\cdot,\cdot)$ denotes the $L^2(0,1)$-inner product.

Hint: multiply the $4th - order$ equation by $v \in W$, integrate both sides over $(0,1)$, and then apply integration by parts twice.

(b) For $I = [a,b]$ an interval, define

$$P_3(I) = \{v : \ v \text{ is a polynomial of degree } \leq 3 \text{ on } I\},$$

i.e. v has the form $v(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for $x \in I$, and $a_i \in R$.

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding (local) basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = 0$, $v(b) = 0$, $v'(b) = 0$, etc.)

(c) Starting from (b), construct a finite-dimensional subspace W_h of W consisting of piecewise-cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding (global) basis functions.
(d) Formulate a FEM for the problem based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition.

• Note: For problems [2] and [3], you can use some software that can help find the basis functions and evaluate the integrals needed for the matrix A (since the evaluation by hand would be more tedious). Or you can leave out this part of the question, or just give the formulas for the basis functions and integrals without carrying out the calculations for obtaining their explicit values (see also Section 3.2 from the textbook by Johnson).