HW #4, 269C Due on Wednesday, May 29

[**1**] Let K be a tetrahedron with vertices a^i, $i = 1, ..., 4$, and let a^{ij} denote the midpoint on the straight line $a^i a^j$, $i < j$. Show that a function $v \in P_2(K)$ is uniquely determined by the degrees of freedom: $v(a^i)$, $v(a^{ij})$, $i, j = 1, ..., 4$, $i < j$. Show that the corresponding finite element space V_h satisfies $V_h \subset C^0(\Omega)$, assuming continuity at all degrees of freedom.

[**2**] Let K be a triangle with vertices a^i, $i = 1, 2, 3$. Suppose that $v \in P_r(K)$ and that v vanishes on the side $a^2 a^3$. Prove that v has the form

$$v(x) = \lambda_1(x) w_{r-1}(x), \quad x \in K,$$

where $w_{r-1} \in P_{r-1}(K)$, and $\lambda_1(x)$ is the affine local basis function corresponding to the node a^1.

(for simplicity, you can assume that K is the reference triangle with vertices (0,0), (0,1) and (1,0), and that the side $a^2 a^3$ is on one of the axes).

[**3**] Let K be a triangle with vertices a^i, $i = 1, 2, 3$, and let a^{ij}, $i < j$, denote the midpoints of the sides of K. Let a^{123} denote the center of gravity of K. Prove that $v \in P_4(K)$ is uniquely determined by the following degrees of freedom

$$\begin{aligned}
&v(a^i), \\
&\frac{\partial v}{\partial x_j}(a^i), \;\; i = 1, 2, 3, \;\; j = 1, 2, \\
&v(a^{ij}), \;\; i, j = 1, 2, 3, \;\; i < j, \\
&v(a^{123}), \;\; \frac{\partial v}{\partial x_j}(a^{123}), j = 1, 2,
\end{aligned}$$

(typo in Exercise 3.8 in the textbook).

Also show that the functions in the corresponding finite element space V_h are continuous, assuming continuity for all degrees of freedom.

[**4**] Consider the PDE (in the distributional sense)

$$-\triangle u + k^2 u = f \quad \text{in } R^n,$$

where k is a constant. Let $s \in R$. Show that, for all $f \in H^s(R^n)$, there exists a unique $u \in H^{s+2}(R^n)$, solution of the PDE, with $k \in R$, $k \neq 0$.

Hint: use the Fourier transform (see handout for notations of Sobolev spaces with arbitrary exponent s).

[**5**] Let $I = [0, h]$ and let $\pi v \in P_1(I)$ be the linear interpolant that agrees with $v \in C^2(I)$ at the end points of I. Using the technique of the proof of Thm. 4.1, prove estimates for $\|v - \pi v\|_{L^\infty(I)}$ and $\|v' - (\pi v)'\|_{L^\infty(I)}$, cf. (1.12) and (1.13).