HW #2. Math 269C (due on Monday, April 29)

[1] Let V be the space considered for the 1D model problem derived from:

$$\text{(D)} \qquad -u'' = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

Construct a finite-dimensional subspace V_h of V consisting of functions which are quadratic on each subinterval I_j of a partition of $I = (0,1)$. How can one choose the parameters to describe such functions ? Find the corresponding basis functions. Then formulate a finite element method for (D) using the space V_h and write down the corresponding linear system of equations in the case of a uniform partition.

[2] Consider the BVP

$$\frac{d^4u}{dx^4} = f, \quad 0 < x < 1, \quad u(0) = u'(0) = u(1) = u'(1) = 0.$$

(a) Show that this problem can be given the following variational formulation: Find $u \in W$ such that

$$(u'', v'') = (f, v), \quad \forall v \in W,$$

where $W = H_0^2(0,1)$ (thus functions in $H^2(0,1)$ with $v(0) = v'(0) = v(1) = v'(1) = 0$).

(b) For $I = [a,b]$ an interval, define

$$P_3(I) = \{v : \ v \text{ is a polynomial of degree } \leq 3 \text{ on } I\},$$

i.e. v has the form $v(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ for $x \in I$, and $a_i \in R$.

Show that $v \in P_3(I)$ is uniquely determined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$. Find the corresponding (local) basis functions (the basis function corresponding to the value $v(a)$ is the cubic polynomial v such that $v(a) = 1$, $v'(a) = 0$, $v(b) = 0$, $v'(b) = 0$, etc.)

(c) Starting from (b) construct a finite-dimensional subspace W_h of W consisting of piecewise-cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding (global) basis functions.

(d) Formulate a FEM for the problem based on the space W_h. Find the corresponding linear system of equations in the case of a uniform partition.

• Note: for the question "Find the corresponding basis functions." in problems [1] and [2], you can use some software that can evaluate the integrals (since the evaluation by hand would be tedious). Or it is fine to leave out this part of the question, or just give the formulas without completely evaluating the integrals.

[3] This question corresponds to the 2D model problem. Find the linear basis functions for the triangle K with vertices at $(0,0)$, $(h,0)$ and $(0,h)$. Show that the corresponding element stiffness matrix is given by

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Using this result, show that the linear system (1.25) of Example 1.1. (textbook) has the stated form (pages 30-31) (there is a typo in the textbook regarding the matrix).