HW #3, Math 269C   Due on Monday, May 8

[**1**] Consider the problem with an inhomogeneous boundary condition,

$$\begin{cases} -\triangle u = f \text{ in } \Omega, \\ u = u_0 \text{ on } \Gamma = \partial\Omega, \end{cases}$$

where f and u_0 are given. Briefly show that this problem can be given the following equivalent formulations:

(V) Find $u \in V(u_0)$ such that $a(u, v) = (f, v)$, $\forall v \in H_0^1(\Omega)$,

(M) Find $u \in V(u_0)$ such that $F(u) \le F(v)$, $\forall v \in V(u_0)$,

where $V(u_0) = \{v \in H^1(\Omega) : \ v = u_0 \text{ on } \Gamma\}$ and F is defined in the usual way, $F(v) = \frac{1}{2}a(v, v) - (f, v)$.

Then formulate a finite element method and prove an error estimate (as in Thm. 1.1, page 24).

Recall: $H_0^1(\Omega) = \{v \in L^2(\Omega),\ \nabla v \in L^2(\Omega)^n,\ v = 0 \text{ on } \partial\Omega\}$, where n is the spatial dimension.

[**2**] (a) Give a weak variational formulation of the problem

$$\frac{d^4u}{dx^4} = f \quad \text{for} \quad 0 < x < 1,$$

$$u(0) = u''(0) = u'(1) = u'''(1) = 0,$$

and show that the assumptions of the Lax-Milgram Lemma are satisfied. Which boundary conditions are essential and which are natural ?

(b) Solve the same problem with the following alternative boundary conditions:

$$u(0) = -u''(0) + \gamma u'(0) = 0, \quad u(1) = u''(1) + \gamma u'(1) = 0,$$

where γ is a positive constant.

[**3**] Give a weak variational formulation of the Robin's problem

$$-\triangle u = f \ \text{ in } \Omega, \quad \gamma u + \frac{\partial u}{\partial n} = g \text{ on } \Gamma,$$

where γ is a constant. When are the assumptions of the Lax-Milgram Lemma satisfied ?

[**4**] Consider the Neumann problem

$$-\triangle u = f \text{ in } \Omega,$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega,$$

$$\int_\Omega u(x)dx = 0.$$

where $f : \Omega \to R$ and $g : \partial\Omega \to R$ satisfy the *compatibility condition*

$$\int_{\Omega} f(x)dx + \int_{\partial\Omega} g(x)d\sigma(x) = 0.$$

(a) Why condition "$\int_{\Omega} u(x)dx = 0$" was added here ? Why do we need the compatibility condition ?

(b) Give a weak variational formulation of the problem, and prove that the conditions of the Lax-Milgram Lemma are satisfied, under the necessary assumptions on f and g that you would specify.